

A FREE STOCHASTIC PARTIAL DIFFERENTIAL EQUATION

YOANN DABROWSKI

ABSTRACT. We get stationary solutions of a free stochastic partial differential equation. As an application, we prove equality of non-microstate and microstate free entropy dimensions under a Lipschitz like condition on conjugate variables, assuming also R^ω embeddability. This includes an N -tuple of q -Gaussian random variables e.g. for $|q|N \leq 0.13$. More generally for $|q|N < 1$ and $|q|\sqrt{N} \leq 0.13$ we prove they still have microstate free entropy dimension N .

INTRODUCTION

In a fundamental series of papers, Voiculescu introduced analogs of entropy and Fisher information in the context of free probability theory. A first microstate free entropy $\chi(X_1, \dots, X_n)$ is defined as a normalized limit of the volume of sets of microstate i.e. matricial approximants (in moments) of the n -tuple of self-adjoints X_i living in a (tracial) W^* -probability space M . Starting from a definition of a free Fisher information [53], Voiculescu also defined a non-microstate free entropy $\chi^*(X_1, \dots, X_n)$, known by the fundamental work [1] to be greater than the previous microstate entropy, and believed to be equal (at least modulo Connes' embedding conjecture). For more details, we refer the reader to the survey [55] for a list of properties as well as applications of free entropies in the theory of von Neumann algebras.

Moreover in case of infinite entropy, two other invariants the microstate and non-microstate free entropy dimensions (respectively written $\delta_0(X_1, \dots, X_n)$ and $\delta^*(X_1, \dots, X_n)$) have been introduced to generalize results known for finite entropy. Surprisingly, Connes and Shlyakhtenko found in [10] a relation between those entropy dimensions and the first L^2 -Betti numbers they defined for finite von Neumann algebras. For instance, for (real and imaginary parts of) generators of (finitely generated) groups, δ^* has been proved in [29] to be equal to $\beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$ (cf. e.g. [27] for L^2 -Betti numbers of groups).

In [46], Dimitri Shlyakhtenko obtained lower bounds on microstate free entropy dimension (motivated by the goal of trying to prove equality with non-microstate free entropy dimension,) in studying the following free stochastic differential equation :

$$X_t^{(i)} = X_0^{(i)} - \frac{1}{2} \int_0^t \xi_s^{(i)} ds + S_t^{(i)}$$

2000 *Mathematics Subject Classification.* 46L54, 60H15.

Key words and phrases. Free Stochastic Partial Differential Equations; Lower bounds on microstate Free Entropy Dimension; Free probability; q -Gaussian variables.

Research partially supported by NSF grants DMS-0555680 and DMS-0900776.

where ξ_s^i is the i -th conjugate variable of $X_s^{(i)}$'s in the sense of [53], $S_t^{(i)}$ a free Brownian motion free with respect to $X_0^{(i)}$. Let us recall that for $(M = W^*(X_1, \dots, X_N), \tau)$, if X_1, \dots, X_n are algebraically free, the i -th partial free difference quotient $\partial_i : L^2(M) \rightarrow L^2(M) \otimes L^2(M)$ is the unique derivation densely defined (on non-commutative polynomials) such that $\partial_j(X_i) = 1_{i=j}1 \otimes 1$. Then the i -th conjugate variable is defined by $\partial_i^*(1 \otimes 1) \in L^2(M)$ if it exists. In [46], this equation was solved in order to get stationary solutions for analytic conjugate variable, and thus this paper proved that in case of analytic conjugate variable if moreover $W^*(X_1, \dots, X_n)$ is R^ω embeddable, then $\delta_0(X_1, \dots, X_n) = \delta^*(X_1, \dots, X_n) = n$. Of course, if we believe in the previous general equality, this should be proved in much more general cases, e.g. for L^2 conjugate variable, i.e. finite Fisher information. The goal of this paper is to prove this equality in an intermediate case, under a Lipschitz like condition on conjugate variables. Let us emphasize our definition does not involve operator Lipschitz functions and is relative to M , but it is nothing but the usual notion of being a Lipschitz function of X (for instance applied by functional calculus) in the one variable case (this is a Sobolev like definition of lipschitzness in the one variable case) :

Definition 1. $(M = W^*(X_1, \dots, X_N), \tau)$ is said to satisfy a *Lipschitz conjugate variable condition* if the partial free difference quotients ∂_i are defined and if the conjugate variables $\partial_i^*1 \otimes 1$ exist in $L^2(M)$ (for all i) and moreover are in the domain of the closure $\bar{\partial}$ of $(\partial_1, \dots, \partial_N)$ with $\bar{\partial}_j \partial_i^*1 \otimes 1 \in M \bar{\otimes} M^{op} \subset L^2(M \otimes M^{op}) \simeq L^2(M \otimes M)$ (von Neumann tensor product).

Let us state a precise result, the main byproduct of our work in this respect (cf. corollary 38) is the following :

Theorem. *Consider $(M = W^*(X_1, \dots, X_N), \tau)$ a R^ω -embeddable finite von Neumann algebra satisfying a Lipschitz conjugate variable condition. Then the microstate entropy dimension $\delta_0(X_1, \dots, X_N) = N$.*

We show in section 4.3 that q -Gaussian variables (introduced in [7]) are a non-trivial instance of non-commutative variables having Lipschitz conjugate variables for small q (e.g. $|q|N \leq 0.13$ thus improving a computation in [46] and proving that δ_0 does not only converge to N for small q but is identically equal to N and thus equal to $\delta^*(X_1, \dots, X_N)$). We actually prove in section 4.4 this is still valid on a slightly larger range of q 's, i.e. as soon as $|q|N < 1$ and $|q|\sqrt{N} \leq 0.13$.

Let us come back to our stochastic differential equation setting. By lack of a theory of "non-commutative Lipschitz functions", we will rather solve a dual Stochastic Partial Differential Equation with the right stationarity property to get this result.

To explain the equation we solve, let us note that if we call $\Phi_s(X) = X_s$ the automorphism we hope being able to build solving the above equation, then Ito formula implies e.g. for any non-commutative polynomials P :

$$\Phi_t(P(X_0)) = \Phi_0(P(X_0)) - \frac{1}{2} \int_0^t \Phi_s(\Delta(P(X_0)))ds + \int_0^t \Phi_s \otimes \Phi_s(\delta(P(X_0)))\#dS_s.$$

We refer the reader to the main text for reminders about free stochastic integration. Here $\delta = (\delta_1, \dots, \delta_n)$ is the free difference quotient, $\Delta = \delta^*\delta$. This is also the equation the author solved in a more recent paper [12] in a much more general context but with more limited applications to microstate free entropy dimensions, because of a lack of control on the von Neumann algebra in which we build the process in this new approach.

Here we will thus rather solve the following dual equation :

$$(1) \quad X_t = X_0 - \frac{1}{2} \int_0^t \Delta(X_s) ds + \int_0^t \delta(X_s) \# dS_s,$$

where δ will be an appropriate extension of the free difference quotient by zero on the free Brownian motion and a corresponding $\Delta = \delta^* \delta$. It is well known in quantum stochastic integration over symmetric Fock space that solving right Hudson Parthasarathy equations instead of left HP equations enables to solve equations in a mild sense (see e.g. [18]) as in the classical Stochastic Differential equation case (see e.g. [13]). This is in order to use those techniques we considered this equation rather than the previous one.

Before describing the content of this paper, let us explain the relation of our work with classical stochastic partial differential equations. There are basically three main approaches to analysing SPDEs: the “martingale (or martingale measure) approach” (cf. [56]), the “semigroup (or mild solution) approach” (cf. [13]) and the “variational approach” (cf. [38]). We will mainly refer to the above monographs instead of the original enormously rich literature. Beyond those mainstream approaches, one should also mention Krylov’s L^p -theory [23] and Kostelenez’s methods [22] using limits of particle systems, and also an old approach for more concrete SPDEs using SDEs in nuclear spaces and distributions (e.g. [50]). Here we will adapt to the free SDE context a part of the semigroup approach using variational techniques. To compare with our work, we thus insist here on those two approaches.

To fix ideas a general SPDE considered in the classical literature is often of the form :

$$dX_t(\xi) = A(t, X_t(\xi), D_\xi X_t(\xi), D_\xi^2 X_t(\xi))dt + B(t, X_t(\xi), D_\xi X_t(\xi))dW_t.$$

Since this will be our main interest, we will mainly focus on the linear time independent case where A is thus a second order differential operator, B a first order one, let say valued in Hilbert-Schmidt operators from the noise space Y (let say W_t is a standard (with covariance Id) cylindrical brownian motion on a Hilbert space Y) to the space H where X_t lives. The linear case was also motivated in the early theory by filtering problems giving rise to linear equations suitable for the variational approach.

Both approaches share the common features of considering SPDEs as SDEs valued in infinite dimensional spaces (usually Hilbert spaces of Sobolev types), using PDE techniques often in an abstract functional analytic setting.

Let us describe first the variational approach, originating from [33],[34], [24] (we refer to [38], and the recent introductory [37] in the coercive case). Usually, solutions are built here by a Galerkin scheme, in first projecting the equation to finite dimensional sub-Hilbert spaces. After this transformation, the equation is an ordinary SDE solved by usual techniques. At this level, estimates (for this approximation) are proved enabling to take a (weak) limit. The equation is first solved in a weak sense, avoiding to require $X_t \in D(A)$. The standard assumption is the so called coercivity condition (also called superparabolic case in [38] when considered for concrete differential operators).

This is roughly written :

$$2\langle x, Ax \rangle + \|B(x)\|_{HS}^2 + \delta \|x\|_U^2 \leq K \|x\|_H^2,$$

where U is another Hilbert space such that $U \subset D(B)$ continuously (often if A is time independent self-adjoint, $-A$ positive $U = D((-A)^{1/2})$ for instance to fix ideas). Having $\delta > 0$ then enables to get a bound on $\|X_t\|_H$ and say $\int_0^t \|X_s\|_U^2 ds$ giving sufficiently many

regularity to get a weak limit such that $B(X_t)$ makes sense and to solve the equation weakly (i.e. after taking scalar products with $y \in D((-A)^{1/2})$ for instance in the self-adjoint case). Unfortunately, the case we are interested in is not coercive, it only satisfies the dissipative condition where $\delta = 0$ above (sometimes called degenerate parabolic case). This equation is enough to guaranty a bound on $\|X_t\|_H$ but nothing more. In this case, the usual method (for instance used in [38] Chapter 4 in a concrete differential operator setting), is to replace B by $(1 - \epsilon)B$ (or A by $(1 + \epsilon)A$) to get a coercive equation and get the bound on $\|X_t\|_U$ necessary to get a weak limit by another technique. In the coercive case, there are also standard ways of getting regularity results (for instance, we assume a dissipative inequality under an overall $(-A)^{1/2}$ for instance again in the self-adjoint positive case, i.e. $-2\langle Ax, Ax \rangle + \|(-A)^{1/2}B(x)\|_{HS}^2 \leq K\|(-A)^{1/2}x\|_H^2$, and not surprisingly deduce from this a bound on $\|(-A)^{1/2}X_t\|_H$, the equation being ideal to apply Gronwall's lemma and get a bound in that way). One thus uses these standard ways of getting regularity via a priori estimates for the approximating equation to get a weak limit.

The semigroup approach uses the semigroup ϕ_t generated by A and rewrites the equation after "variation of constants", and thus looks for a so-called mild solution, i.e. a solution of :

$$X_t = \phi_t(X_0) + \int_0^t \phi_{t-s}(B(X_s)dW_s).$$

Then the goal is to use regularization properties of this semigroup to solve this equation. For instance, to solve non-linear equations with only continuous coefficients for B , one can use compactness of the semigroup and use compactness arguments (and get a stochastically weak solution, i.e. not adapted to the filtration of the Brownian motion. Note we use in this paper only the word weak in its PDE sense as in [13]). In more standard assumptions, the semigroup is only assumed analytic, or with generator a variational or a self-adjoint operator. We will be mainly interested in the semigroup approach under the same assumptions as in the variational approach. Indeed, in our free SDE setting, it is not quite clear what kind of Galerkin's method could make us recover an ordinary free SDE setting. Moreover as we will see, we will use extensively really weak notions of being a mild solution we will call ultramild as a crucial tool to get results on really weak assumptions. Anyways, the interest of the semigroup approach for us lies in the fact it replaces Galerkin's method by a fixed point argument (for contractions) under the same coercivity assumption. Then, we can again prove a priori estimates to extend this to the degenerate parabolic case we will be interested in (since only this case can give stationary solutions in our examples).

Let us now finally describe the content of this article. In Section 1, we solve a really general stochastic partial differential equation (formally of the form (1)) with much less restrictive assumptions on δ , Δ . We find natural assumptions to get two kinds of solutions we will call mild and ultramild solutions, this second really weak sense of getting a solution has never been considered, to the best of our knowledge, in previously quoted contexts. These conditions are natural analogs of the dissipativity condition above (in case of ultramild solutions) and the dissipativity condition under $(-A)^{1/2}$ (to get regularity conditions and for us mild solutions). We have also to include in these conditions general compatibility assumption trivially checked in our main example.

In Section 2, we prove that we can check our assumptions to get mild solutions in the free difference quotient case with a Lipschitz conjugate variable type assumption as explained

above. Actually, we also check that we can slightly relax the coassociativity property of the free difference quotient crucially used here (even in this weakened form, for instance applicable to liberation gradient derivations of [54]). The crucial issue here is to prove non-trivial domain properties of δ, Δ which are usually checked classically using general regularity results of PDEs not available in our non-commutative context. One of the crucial tools here is (a variant of) an easy boundedness criteria for $1 \otimes \tau \circ \delta$ found by the author in [11] (cf. Lemma 30 *infra.* and compare with [11, lemma 10]).

In Section 3, we prove that as soon as we stick to our case of main interest of a derivation and the corresponding divergence form operator, it suffices to check $\|X_t\|_2 = \|X_0\|_2$ in order to prove any ultramild solution to be stationary, as we want in order to get lower bounds on microstate free entropy dimension. Especially, this is always true if we can get a mild solution, and this is really likely why ultramild solutions were never considered before, without at least an isometry property, solving those equations is not such useful.

In Section 4, we explain our main application about computation of microstate free entropy dimension under Lipschitz conjugate variable assumption. In section 4.3, as we said above, we explain the concrete example of q -Gaussian variables, after several general preliminaries gathered in section 4.2. Here the proof of Lipschitz conjugate variable relies heavily on Bożejko's analog of Haagerup's Inequalities [5] and on inequalities of our section 2. We also consider in section 4.4 how one can use a non-coassociative derivation to compute microstate free entropy dimension of q -Gaussian variables in a slightly less small range of q 's. This motivates the extra work in this context in section 2. Of course it is possible that a better understanding of combinatorial properties of those examples may give more extended ranges of q 's with the same free SDE techniques. Finally, in section 4.5, we explain how hard it is to get stationary solutions in an example of derivations on group von Neumann algebras coming from group cocycles valued in the left regular representation, case also considered in [46]. Here coassociativity like assumptions are not available to get "easily" mild solutions, this is why we were motivated in being able to get solutions in a really general sense like ultramild solutions under somehow an automatically verified assumption. Indeed, in such a concrete example one can easily find a necessary and sufficient condition for getting $\|X_t\|_2 = \|X_0\|_2$. However, it is expressed in terms of conservativity of a classical Markov process, well known to be hard to check. This is not such surprising since unitarity properties of left Hudson-Parthasarathy equations are also expressed in terms of conservativity of quantum Markov processes (see e.g. the survey [17]). Of course, the occurrence of a classical process is only explained by our special example on groups, anyone interested in such a criteria for more general processes may be able to generalize this to a general case using conservativity of an appropriate quantum Markov process. However since any useful (easy to check) sufficient condition for proving this conservativity is not really available (even in HP case) beyond conditions really similar to those of our section 2 to get mild solutions (cf. [8]), we don't enter in this general question here. Let us conclude with two remarks. Having in mind those similarities with questions of unitarity of solutions of Hudson-Parthasarathy equations, we can wonder whether a duality theory analogous to (Journe) duality of left and right HP equations could be developed in our context. In the other direction, one may wonder whether an ultramild like definition of a solution may be useful for right HP equations (e.g. in order to solve them under weaker conditions expressed in terms of conservativity assumptions similar

to left HP equations) or whether the new approach of [12] could be translated in the context of left HP equations.

Acknowledgments The author would like to thank D. Shlyakhtenko and P. Biane for plenty of useful discussions, A. Guionnet and an anonymous referee for plenty of useful comments helping him improve the exposition of previous versions of this text.

1. A GENERAL STOCHASTIC DIFFERENTIAL EQUATION WITH UNBOUNDED COEFFICIENTS

Let M_0 be a W^* -probability space (with separable predual), $S_t^{(i)}$ ($i \in \mathbb{N}$) a family of free Brownian motions. Consider $M = M_0 \star W^*(S_t^{(i)})$ the free product of W^* -probability spaces (so that $S_t^{(i)}$ are free with M_0 inside M) and consider finally the natural filtration $M_s = M_0 \star W^*(S_t^{(i)}, t \leq s)$. As a side remark, note we always use scalar products linear in the second variable.

In this part, we will be interested in the following equation :

$$(2) \quad X_t = X_0 - \frac{1}{2} \int_0^t \Delta(X_s) ds + \int_0^t \delta(X_s) \# dS_s,$$

where $\Delta : L^2(M) \rightarrow L^2(M)$ and $\delta : L^2(M) \rightarrow L^2(M) \otimes L^2(M) \oplus \mathbb{N}$ are closed densely defined operators and keeping invariant for $t \in [0, T]$ $L^2(M_t)$ (resp. sending it to $L^2(M_t \otimes M_t) \oplus \mathbb{N}$ and with the analog property for its adjoint. See subsection 1.2 for a definition of Stochastic integral). The sense in which we will solve this equation will be made precise in the 3 following sections : the first will deal with some miscellaneous results about stochastic integration in our context, the second will introduce stochastic convolution, the key tool to define mild solutions and the third one will prove in the free Brownian case some well-known (in the classical Brownian motion case) relations between mild and strong solutions, and introduce ultramild solutions (the three kinds of solutions we will be interested in getting). Let us right now state the two class of assumptions we will need to get mild (resp. ultramild) solutions in the last subsection of this section¹. To state a slightly more general result (for instance enabling us later to reach $|q|N < 1$ instead of $|q| \leq 1/((1 + \sqrt{2})N + 2)$ in the q -Gaussian variable case), we also consider given another operator $\tilde{\delta}$ satisfying the assumptions for δ , i.e $\tilde{\delta} : L^2(M) \rightarrow L^2(M) \otimes L^2(M) \oplus \mathbb{N}$ are closed densely defined operators keeping invariant the corresponding filtrations.

We fix a few notations before stating the assumption. We will write $U \# (S_t - S_s) = \sum_{i=0}^{\infty} U^{(i)} \# (S_t^{(i)} - S_s^{(i)})$ the Hilbert space isomorphism between the infinite direct sum of coarse correspondences $L^2(M_s) \otimes L^2(M_s) \oplus \mathbb{N}$ where $U^{(i)} \# (S_t^{(i)} - S_s^{(i)})$ is the linear isomorphism extending $a \otimes b \# (S_t^{(i)} - S_s^{(i)}) = a(S_t^{(i)} - S_s^{(i)})b$, for $a, b, c \in M_s$. Likewise, for 3-fold tensor products, we write $(a \otimes b \otimes c) \#_1 (S_t^{(i)} - S_s^{(i)}) = a(S_t^{(i)} - S_s^{(i)})b \otimes c$, $(a \otimes b \otimes c) \#_2 (S_t^{(i)} - S_s^{(i)}) =$

¹Note we will always write $A \circ B$ the closure of the composition of two closed operators if possible, and the usual composition if they are not closed, without risk of confusion. Sometimes we will even write AB for the same object.

$a \otimes b(S_t^{(i)} - S_s^{(i)})c$ and their corresponding L^2 extensions. On a direct sum, we write $Diag(A_i)$ the operator acting diagonally, e.g. $Diag(A_i)(b_1 \oplus b_2) = A_1 b_1 \oplus A_2 b_2$.

The main assumption (useful to get mild solutions) will be called $\Gamma_1(\omega, C)$:

$$\Gamma_1(\omega, C) \left\{ \begin{array}{l} a) \quad \Delta \text{ is a positive self-adjoint operator, } \eta_\alpha = \frac{\alpha}{\alpha + \Delta}. \\ b) \quad D(\Delta^{1/2}) \subset D(\delta), \\ b') \quad D(\delta \circ \Delta) \subset D(\delta) \\ c) \quad \text{for any } x \in D(\Delta^{1/2}) \text{ we have : } -\|\Delta^{1/2}x\|_2^2 + \|\delta(x)\|_2^2 \leq \omega\|x\|_2^2 \\ c') \quad \text{for any } x \in D(\Delta^{1/2}) : -\|\Delta\eta_\alpha^{1/2}x\|_2^2 + Re\langle \delta(\Delta\eta_\alpha(x)), \delta(x) \rangle \leq \omega\|\Delta^{1/2}\eta_\alpha^{1/2}x\|_2^2 \\ d) \quad \text{There exists a closed densely defined positive operator} \\ \Delta^\otimes =: Diag(\Delta_i^\otimes) : (L^2(M) \otimes L^2(M)) \oplus \mathbb{N} \rightarrow (L^2(M) \otimes L^2(M)) \oplus \mathbb{N} \text{ (acting} \\ \text{diagonally with respect to the direct sum and keeping invariant,} \\ \text{for any } t, L^2(M_t \otimes M_t) \oplus \mathbb{N} \text{ and) such that for any} \\ U \in L^2(M_s) \otimes L^2(M_s): U \# (S_t^{(i)} - S_s^{(i)}) \in D(\Delta) \text{ if } U \in D(\Delta_i^\otimes) \\ \text{and } \Delta(U \# (S_t^{(i)} - S_s^{(i)})) = \Delta_i^\otimes(U) \# (S_t^{(i)} - S_s^{(i)}) \\ \text{Moreover } \delta(U \# (S_t^{(i)} - S_s^{(i)})) \text{ is orthogonal to } L^2(M_s \otimes M_s) \\ d') \quad D(\Delta^{1/2}) \subset D(\tilde{\delta}), D(\Delta^{1/2}) \text{ a core for } \tilde{\delta}, D(\tilde{\delta} \circ \Delta) \subset D(\tilde{\delta}) \\ \text{and } \tilde{\delta}(U \# (S_t^{(i)} - S_s^{(i)})) \text{ is orthogonal to } L^2(M_s \otimes M_s) \ni U, \\ \text{and we assume we have a closed densely defined} \\ \tilde{\delta}^\otimes := \tilde{\delta}^{\otimes 1} \oplus \tilde{\delta}^{\otimes 2}, \tilde{\delta}^{\otimes i} : (L^2(M^{\otimes 2})) \oplus \mathbb{N} \rightarrow (L^2(M^{\otimes 3})) \oplus \mathbb{N}^2 \\ \tilde{\delta}^{\otimes i}((x_j)_j) =: (\tilde{\delta}_{l,j}^{\otimes i}(x_j))_{(l,j)}, \tilde{\delta}_{l,j}^{\otimes i} : (L^2(M)^{\otimes 2}) \rightarrow (L^2(M)^{\otimes 3}) \\ \text{(keeping invariant, for any } t, \text{ the filtration induced by } M_t \text{ and) such that for any} \\ U \in L^2(M_s) \otimes L^2(M_s): U \# (S_t^{(j)} - S_s^{(j)}) \in D(\tilde{\delta}_l) \text{ if } U \in D(\tilde{\delta}_{l,j}^{\otimes 1} \oplus \tilde{\delta}_{l,j}^{\otimes 2}) \\ \text{and } \tilde{\delta}_l(U \# (S_t^{(j)} - S_s^{(j)})) = \sum_{i=1}^2 \tilde{\delta}_{l,j}^{\otimes i}(U) \#_i (S_t^{(j)} - S_s^{(j)}) \\ e) \quad D(\tilde{\delta} \circ \Delta) \subset D(\Delta^\otimes \circ \tilde{\delta}) \\ f) \quad \text{There exists a bounded operator } \mathcal{H} \text{ on } L^2(M \otimes M) \oplus \mathbb{N} \\ \text{keeping invariant for any } s, L^2(M_s \otimes M_s) \oplus \mathbb{N} \text{ with } \|\mathcal{H}\| \leq C^{1/2} \text{ (} C \geq 1 \text{)} \\ \text{such that for any } x \in D(\tilde{\delta} \circ \Delta): \\ \Delta^\otimes \circ \tilde{\delta}(x) - \tilde{\delta} \circ \Delta(x) = \mathcal{H}(\tilde{\delta}(x)) \\ g) \quad D(\tilde{\delta}) = D(\delta) =: \mathcal{D}, \forall x \in \mathcal{D}, C^{-1/2}\|\tilde{\delta}(x)\|_2 \leq \|\delta(x)\|_2 \leq C\|\tilde{\delta}(x)\|_2, \\ \text{thus } D(\tilde{\delta} \circ \Delta) = D(\delta \circ \Delta) \\ h) \quad D(\Delta) \subset D(\Delta^{\otimes 1/2} \circ \tilde{\delta} \oplus \tilde{\delta} \oplus \tilde{\delta}^\otimes \delta) \text{ and for } x \in D(\Delta) \\ \|\tilde{\delta}^\otimes \delta(x)\|_2^2 \leq \|\Delta^{\otimes 1/2} \circ \tilde{\delta}(x)\|_2^2 + C\|\tilde{\delta}(x)\|_2^2 \end{array} \right.$$

We will write ϕ_t the semigroup exponentiating $-1/2\Delta$ and ϕ_t^\otimes the semigroup associated to $-1/2\Delta^\otimes$.

In most cases we will be interested in the case $\tilde{\delta} = \delta$ in which case assumptions g, h are automatic (using c for h , d' also almost follows from b, d and the remaining part will be easy to check in the applications we are interested in).

In the applications we have in mind, the strong assumptions are the two previous ones (e, f) , the other ones being automatically verified and just important in this general setting. Without (b', c', d', e, f, g, h) , we call $\Gamma_0(\omega)$ the set of assumptions a, b, c, d . The reader will see later this is the right set of assumptions to get a really weak form of solution we will call ultramild solution. Obtaining this kind of solution will thus be almost automatic as we said, since only e and f are constraining assumptions.

General ideas and strategy

With those notation fixed and before entering into technical details, let us explain the intuition behind our results (in the case $\tilde{\delta} = \delta$, the general case is a slight extension following an idea of [8], this will be useful only to improve results in the q-Gaussian case). In our general setting here, the proofs will follow closely the classical case, and therefore the intuition is basically the same, namely, since we want to solve SDEs with unbounded coefficient, with Δ a kind of divergence form operator (as we will consider in the next part) the corresponding semigroup is regularizing, and we want to use this. That's why we introduce mild solutions. As explained by various equivalences in section 2.1.3, the difference with strong solutions is only related to the domain in which we want to build the solution, we only require being in the domain of $\Delta^{1/2}$ (or even δ) for mild solutions, and as soon as it is in the domain of Δ , a mild solution is a strong solution, the converse being always true. The idea behind ultramild solutions is slightly trickier. Let us explain it in saying $\phi_{t-s}^{\otimes} \circ \delta$ may have a much huger domain again than $\Delta^{1/2}$ and we want to use this regularization effect to have solutions with almost no conditions. Indeed, conditions e) and f) above will be really hard to check even with strong conditions (section 2.2), that's why we want to have solutions in a sense as general as possible. We can also say that the current section somehow takes natural analogs of classical assumptions in the non-commutative case and check we can work with them.

It is maybe also useful to have several ideas in mind, and first how those conditions will appear in a really natural way in the proof. To get an estimate on $\|X_t\|_2^2$, or $\|\Delta^{1/2}(X_t)\|_2^2$ (first on an approximation of the solution, in the spirit of moving from a degenerate parabolic case to a superparabolic case), the common idea is to differentiate, and try to apply Gronwall's lemma. Conditions c) (called dissipativity in the classical case) and f) above corresponds exactly to what we want, in order to apply this lemma respectively in those cases. The second idea is that if we replace δ by $(1 - \epsilon)\delta$ the equation is much easier to solve, it is of superparabolic type (instead of degenerate parabolic type, said otherwise this gives a kind of coercivity, see [38] for a presentation of this point in a more concrete setting but more clearly than in [13]), and first, in this case, there will be a Picard iteration argument to solve it, second we win something in terms of domains, assumption c is enough to bound $\|\Delta^{1/2}(X_t^\epsilon)\|_2^2$, assumption f to get a bound on $\|\Delta(X_t^\epsilon)\|_2^2$ (those bounds diverging in ϵ , of course). Anyways this will enable us to have respectively strong or mild solutions of an approximating equations converging to a solution of our equation, even if the solution without ϵ will be only a mild or ultramild solution (note that in the case $\delta \neq \tilde{\delta}$ we will lose the strong solution property but keep mild solutions, hopefully we don't use this improvement in terms of getting a strong solution). Somehow, to get later in section 3.3 stationarity of the equation, this will be much more crucial to have an approximation by a mild solution than an ultramild solution, since Ito formula, already tricky to apply for mild solutions, seems to be completely unusable for ultramild solutions. Moreover there is the general idea

that if you get a bound of $\|\Delta^{1/2}(X_t^\epsilon)\|_2^2$ (uniform in ϵ), you can get a Cauchy condition in $\|\cdot\|_2$ norm and thus norm convergence, but however in general we will work only with weak convergence. Finally, we will also show in section 1.3 that mild solutions are also weak solutions, in a usual duality sense of weak solutions, however, we don't have an analog for ultramild solutions. Thus we will also introduce a notion of ultraweak solution, mainly to get unicity results in applying Laplace transforms techniques, our general result will be "there exists a unique ultraweak solutions which is also an ultramild solution", and limit of mild solutions of an approximating equation.

Before starting, we sum up the content of the next sections. In section 1.1, we give miscellaneous definitions and results moving almost commutations properties of our assumptions to stochastic integrals. In section 1.2, we prove an integration by parts formula for a stochastic convolution we introduce. We avoid proving a free variant of the usually used stochastic Fubini Theorem in using ad hoc proofs in our really special case. In section 1.3 we introduce our different kinds of solutions and prove relations between them (explained above). Section 1.4 contains our general theorem.

Let us prove right now an easy consequence of our main assumptions (Note we often use the bound $\|\Delta\eta_\alpha\| \leq 2\alpha$ coming from $\Delta\eta_\alpha = \alpha(1 - \eta_\alpha)$):

Lemma 2. *Assume $\Gamma_1(\omega, C)$. Then, there exists for any $\alpha \in (0, \infty)$ bounded operators \mathcal{H}_α from the graph of $\Delta^{1/2}: G(\Delta^{1/2}) \subset L^2(M)^{\oplus 2}$ to $L^2(M \otimes M) \oplus \mathbb{N}$ sending, for s , $L^2(M_s)^{\oplus 2}$ to $L^2(M_s \otimes M_s) \oplus \mathbb{N}$ with $\|\mathcal{H}_\alpha\| \leq \max(1, \sqrt{\omega})C$ such that for any $x \in D(\Delta^{1/2})$ (if we write $\eta_\alpha = \frac{\alpha}{\alpha + \Delta}$ and the analog $\eta_\alpha^\otimes = \frac{\alpha}{\alpha + \Delta^\otimes}$):*

$$\Delta^\otimes \eta_\alpha^\otimes \circ \tilde{\delta}(x) - \tilde{\delta} \circ \Delta\eta_\alpha(x) = \mathcal{H}_\alpha(x \oplus \Delta^{1/2}(x)).$$

Moreover, for each $x \in D(\Delta^{1/2})$, $\mathcal{H}_\alpha(x \oplus \Delta^{1/2}(x))$ converges in L^2 to $\mathcal{H}(\tilde{\delta}(x))$ when $\alpha \rightarrow \infty$.

Finally, there is also a bounded $\tilde{\mathcal{H}}_\alpha$, with $\|\tilde{\mathcal{H}}_\alpha\| \leq \frac{C}{\alpha}\sqrt{\omega + 2\alpha}$ and the same invariance of filtration properties, such that for any $x \in D(\tilde{\delta})$:

$$\tilde{\delta}\eta_\alpha(x) - \eta_\alpha^\otimes \tilde{\delta}(x) = \tilde{\mathcal{H}}_\alpha(x),$$

Proof. Let \mathcal{H} be given by assumption f . Let us define \mathcal{H}_α . For $x \in D(\Delta^{1/2})$, $\eta_\alpha(x) \in D(\Delta^{3/2}) \subset D(\tilde{\delta} \circ \Delta)$, thus applying the equation for \mathcal{H} in f , we get:

$$\eta_\alpha^\otimes \Delta^\otimes \circ \tilde{\delta}\eta_\alpha(x) - \eta_\alpha^\otimes \tilde{\delta} \circ \Delta\eta_\alpha(x) = \eta_\alpha^\otimes \mathcal{H}(\tilde{\delta}\eta_\alpha(x)).$$

Thus multiplying by $\frac{1}{\alpha}$ and using the definition of the resolvent, and using $\alpha(1 - \eta_\alpha) = \Delta\eta_\alpha$, we get:

$$\tilde{\delta}\eta_\alpha(x) - \eta_\alpha^\otimes \tilde{\delta}(x) = \frac{1}{\alpha} \eta_\alpha^\otimes \mathcal{H}(\tilde{\delta}\eta_\alpha(x)),$$

Especially defining $\tilde{\mathcal{H}}_\alpha = \frac{1}{\alpha} \eta_\alpha^\otimes \mathcal{H} \tilde{\delta}\eta_\alpha$, we get the last statement since (by assumptions f, g, c) $\|\tilde{\mathcal{H}}_\alpha\| \leq \frac{C^{1/2}}{\alpha} \|\tilde{\delta}\eta_\alpha\| \leq \frac{C}{\alpha} \sqrt{\omega + 2\alpha}$ and moreover by d') $D(\Delta^{1/2})$ is a core for $\tilde{\delta}$.

Moreover we also deduce:

$$\Delta^\otimes \eta_\alpha^\otimes \tilde{\delta}(x) = -\frac{1}{\alpha} \Delta^\otimes \eta_\alpha^\otimes \mathcal{H}(\tilde{\delta}\eta_\alpha(x)) + \tilde{\delta}\Delta\eta_\alpha(x) + \mathcal{H}(\tilde{\delta}\eta_\alpha(x)),$$

thus equivalently

$$\Delta^\otimes \eta_\alpha^\otimes \tilde{\delta}(x) - \tilde{\delta}\Delta\eta_\alpha(x) = \eta_\alpha^\otimes \mathcal{H}(\tilde{\delta}\eta_\alpha(x)).$$

This suggests $\mathcal{H}_\alpha(x \oplus \Delta^{1/2}(x)) = \eta_\alpha^\otimes \mathcal{H}(\tilde{\delta}\eta_\alpha(x))$. In that way, the equation is verified, the stability properties come from the assumptions and using property c) of Γ_1 , we get :

$$\|\mathcal{H}_\alpha(x \oplus \Delta^{1/2}(x))\|_2^2 \leq C \|\tilde{\delta}\eta_\alpha(x)\|_2^2 \leq C^2 (\|\Delta^{1/2}(x)\|_2^2 + \omega \|x\|_2^2).$$

Thus we get the bound on $\|\mathcal{H}_\alpha\|$, and

$$\begin{aligned} \|\mathcal{H}_\alpha(x \oplus \Delta^{1/2}(x)) - \mathcal{H}(\tilde{\delta}(x))\|_2 &\leq \|\eta_\alpha^\otimes \mathcal{H}(\tilde{\delta}(\eta_\alpha - 1)(x))\|_2 + \|(\eta_\alpha^\otimes - 1)\mathcal{H}(\tilde{\delta}(x))\|_2 \\ &\leq C \|\tilde{\delta}(\eta_\alpha - 1)(x)\|_2 + \|(\eta_\alpha^\otimes - 1)\mathcal{H}(\tilde{\delta}(x))\|_2, \end{aligned}$$

and the right hand side goes to zero using again assumption c). \square

1.1. Stochastic integration in presence of δ and Δ . Following [2] (except for the value in $L^2(M \otimes M) \oplus \mathbb{N}$ instead of $L^2(M \otimes M)$ of bi-processes), we write $\mathcal{B}_2^a([0, T])$ (for processes on $[0, T]$) the completion of the space of simple adapted (with respect to the algebraic direct sum $(M_t \otimes M_t) \oplus \mathbb{N}$) bi-processes in the following norm :

$$\|U\|_{\mathcal{B}_2^a([0, T])} = \left(\int_0^T \|U_s\|_{L^2(\tau \otimes \tau)}^2 ds \right)^{1/2}.$$

Let us remark that this space can also be seen as a subspace of $L^2([0, T], L^2(\tau \otimes \tau) \oplus \mathbb{N})$ (defined, say, in Bochner's sense) and we will always see it as such a subspace.

Then, recall that the map $U \mapsto \int_0^T U_s \# dS_s = \sum_{j=1}^\infty \int_0^T U_s^{(j)} \# dS_s^{(j)}$ is an isometric linear extension from $\mathcal{B}_2^a([0, T])$ to $L^2(M, \tau)$ of the usual map, sending, for $a, b \in M_s$, $a \otimes b 1_{[s, t]}$ seen in the i -th component of the direct sum to $a(S_t^{(i)} - S_s^{(i)})b$. Thus, we can remark for further use that weak convergence in $L^2(\tau)$ of a sequence of stochastic integrals $\int_0^T U_s^n \# dS_s$ is equivalent to weak convergence of its argument U_s^n in $L^2([0, T], L^2(\tau \otimes \tau) \oplus \infty)$.

Analogously, one can consider $\mathcal{B}_2^{a(\otimes 3)}([0, T])$ for processes $(M_t \otimes M_t \otimes M_t) \oplus (\{1, 2\} \times \mathbb{N}^2)$ -adapted and define $\int_0^T U_s \#_i dS_s \in L^2(M \otimes M) \oplus \mathbb{N}$ ($i = 1, 2$) in extending the definition for $a, b, c \in M_s$, $a \otimes b \otimes c 1_{[s, t]}$ seen in the $1, j, i$ -th component of the direct sum $\int_0^T a \otimes b \otimes c 1_{[s, t]}(u) \# dS_u = a(S_t^{(i)} - S_s^{(i)})b \otimes c$ in the j -th component, and when seen in the $2, i, j$ -th component $\int_0^T a \otimes b \otimes c 1_{[s, t]}(u) \# dS_u = a \otimes b(S_t^{(j)} - S_s^{(j)})c$ in the i -th component.

We will write $\mathcal{B}_{2, \delta \circ \Delta^\beta}^a([0, T])$ (for $\beta \in \{0, 1/2, 1\}$, resp. $\mathcal{B}_{2, \Delta^\beta}^a([0, T])$ for $\beta \in \{1/2, 1, 3/2\}$) the completion with respect to the following norms of what we will call $\delta \circ \Delta^\beta$ -simple adapted processes (resp. Δ^β -simple adapted processes), i.e. processes of the form $X = \sum_{j=1}^M X_j 1_{[t_j, t_{j+1})}$ with $X_j \in D(\Delta^\beta) \cap \bigcap_{b=0}^{2\beta} D(\delta \circ \Delta^{b/2})$ (resp. $X_j \in D(\Delta^\beta)$) :

$$\begin{aligned} \|X\|_{\mathcal{B}_{2, \delta \circ \Delta^\beta}^a} &= \left(\int_0^T \sum_{b=0}^{2\beta} \|\delta \circ \Delta^{b/2} X_s\|_{L^2(\tau \otimes \tau) \oplus \mathbb{N}}^2 ds + \sum_{b=0}^{2\beta} \|\Delta^{b/2} X_s\|_{L^2(\tau)}^2 ds \right)^{1/2} \\ (resp. \|X\|_{\mathcal{B}_{2, \Delta^\beta}^a} &= \left(\int_0^T \sum_{b=0}^{2\beta} \|\Delta^{b/2} X_s\|_{L^2(\tau)}^2 ds \right)^{1/2} .) \end{aligned}$$

Of course, using g , one gets the same spaces if we replace δ by $\tilde{\delta}$. Assuming $\Gamma_0(\omega)$ (especially condition b), we have clearly continuous embeddings $\mathcal{B}_{2, \delta \circ \Delta^\beta}^a([0, T]) \rightarrow L_a^2([0, T], L^2(M))$

(space of adapted processes) $\mathcal{B}_{2,\Delta^\beta}^a([0, T]) \rightarrow L_a^2([0, T], L^2(M))$, $\mathcal{B}_{2,\delta \circ \Delta^\beta}^a([0, T]) \rightarrow \mathcal{B}_{2,\delta \circ \Delta^{\beta'}}^a([0, T])$ for $\beta' \leq \beta$, $\beta, \beta' \in \{0, 1/2, 1\}$ and $\mathcal{B}_{2,\delta \circ \Delta^\beta}^a([0, T]) \rightarrow \mathcal{B}_{2,\Delta^{\beta'}}^a$ for $\beta' \leq \beta$, $\beta, \beta' \in \{1/2, 1\}$, $\mathcal{B}_{2,\Delta^{\beta+1/2}}^a([0, T]) \rightarrow \mathcal{B}_{2,\delta \circ \Delta^\beta}^a([0, T])$ for $\beta \in \{0, 1/2\}$ (using assumption c).

From the assumptions on δ and Δ , we remark that we can see for any $X_s \in \mathcal{B}_{2,\delta \circ \Delta^\beta}^a([0, T])$, $\delta \circ \Delta^\beta X_s$ as an element of \mathcal{B}_2^a . Finally, let us note that if B bounded operator on $L^2(M_s \otimes M_s) \oplus \mathbb{N}$, keeping invariant, for any t , $L^2(M_t \otimes M_t) \oplus \mathbb{N}$, and if $U_s \in \mathcal{B}_2^a([0, T])$, then $B(U_s) \in \mathcal{B}_2^a([0, T])$.

The following lemma is the goal of these definitions :

Lemma 3. Assume $\Gamma_1(\omega, C)$ for (i) and $\Gamma_0(\omega)$ for (ii) and (iii).

- (i) Let $X_s \in \mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$ then we have $\eta_\alpha(X_s) \in \mathcal{B}_{2,\delta \circ \Delta}^a([0, T])$, $\Delta^{\otimes} \eta_\alpha^{\otimes} \tilde{\delta}(X_s), \mathcal{H}_\alpha(X_s \oplus \Delta^{1/2}(X_s)) \in \mathcal{B}_2^a([0, T])$ and $t \leq T$:

$$\int_0^t \Delta^{\otimes} \eta_\alpha^{\otimes} \tilde{\delta}(X_s) \# dS_s = \int_0^t \tilde{\delta} \circ \Delta(\eta_\alpha(X_s)) \# dS_s + \int_0^t \mathcal{H}_\alpha(X_s \oplus \Delta^{1/2}(X_s)) \# dS_s.$$

If $X_s \in \mathcal{B}_{2,\Delta^{3/2}}^a([0, T])$, then $\delta^{\otimes} \delta(X_s) = (\delta_{i,j}^{\otimes k} \delta_j(X_s))_{k,i,j} \in \mathcal{B}_2^{a(\otimes 3)}([0, T])$, $\int_0^t \delta(X_s) \# dS_s \in D(\tilde{\delta})$ and we have the equation :

$$\tilde{\delta} \int_0^t \delta(X_s) \# dS_s = \int_0^t \tilde{\delta}^{\otimes} \delta(X_s) \# dS_s$$

- (ii) Likewise, for any $U_s \in \mathcal{B}_2^a([0, T])$ and $t > 0, \tau \leq T$: $\eta_\alpha^{\otimes}(U_s), \phi_t^{\otimes}(U_s) \in \mathcal{B}_2^a([0, T])$, $\Delta^{\otimes} \eta_\alpha^{\otimes}(U_s), \Delta^{\otimes} \phi_t^{\otimes}(U_s) \in \mathcal{B}_2^a([0, T])$ (and assuming d', g we have also, $\tilde{\delta}^{\otimes} \eta_\alpha^{\otimes}(U_s) = (\tilde{\delta}_{i,j}^{\otimes k} \eta_\alpha^{\otimes}(U_s^j))_{k,i,j} \in \mathcal{B}_2^{a(\otimes 3)}([0, T])$), and we have :

$$\begin{aligned} \eta_\alpha(\int_0^\tau U_s \# dS_s) &= \int_0^\tau \eta_\alpha^{\otimes}(U_s) \# dS_s, \quad \Delta \eta_\alpha(\int_0^\tau U_s \# dS_s) = \int_0^\tau \Delta^{\otimes} \eta_\alpha^{\otimes}(U_s) \# dS_s, \\ \phi_t(\int_0^\tau U_s \# dS_s) &= \int_0^\tau \phi_t^{\otimes}(U_s) \# dS_s, \quad \Delta \phi_t(\int_0^\tau U_s \# dS_s) = \int_0^\tau \Delta^{\otimes} \phi_t^{\otimes}(U_s) \# dS_s, \\ \tilde{\delta} \eta_\alpha(\int_0^\tau U_s \# dS_s) &= \int_0^\tau \tilde{\delta}^{\otimes} \eta_\alpha^{\otimes}(U_s) \# dS_s \quad \text{if } d', g \text{ also hold,} \end{aligned}$$

Finally, for any $W \in L^2(M_t \otimes M_t) \oplus \mathbb{N}$, any $V = (\int_t^\tau U_s \# dS_s)$, $t \leq \tau$, with $V \in D(\delta)$, $\langle W, \delta(V) \rangle = 0$.

- (iii) For $U_s \in \mathcal{B}_2^a([0, T])$, $\int_0^T U_s \# dS_s \in D(\Delta^{1/2})$ if and only if $U_s \in D(\Delta^{\otimes 1/2})$ for almost every s and $\int_0^T ds \|\Delta^{\otimes 1/2} U_s\|_2^2 < \infty$. In this case $\Delta^{1/2} \int_0^T U_s \# dS_s = \int_0^T \Delta^{\otimes 1/2}(U_s) \# dS_s$. If d', g also hold, then for any U_s with $\tilde{\delta}^{\otimes}(U_s) \in \mathcal{B}_2^{a(\otimes 3)}([0, T])$ (e.g. for $U_s = \delta(X_s)$, for $X_s \in \mathcal{B}_{2,\Delta}^a([0, T])$ if h holds), we have $\int_0^T U_s \# dS_s \in D(\tilde{\delta})$ and $\tilde{\delta} \int_0^T U_s \# dS_s = \int_0^T \tilde{\delta}^{\otimes}(U_s) \# dS_s$.

Proof. First of all, the statements about $\eta_\alpha^{\otimes}(U_s), \phi_t^{\otimes}(U_s) \in \mathcal{B}_2^a([0, T])$, $\Delta^{\otimes} \eta_\alpha^{\otimes}(U_s), \Delta^{\otimes} \phi_t^{\otimes}(U_s) \in \mathcal{B}_2^a([0, T])$ and $\Delta^{\otimes} \eta_\alpha^{\otimes} \tilde{\delta}(X_s) \in \mathcal{B}_2^a$ follow from the remark before the lemma, since e.g. $\|\Delta^{\otimes} \eta_\alpha^{\otimes}\| \leq 2\alpha$. Assuming d', g , the same is true for $\tilde{\delta}^{\otimes} \eta_\alpha^{\otimes}(U_s) \in \mathcal{B}_2^{a(\otimes 3)}([0, T])$.

If X_s is a $\Delta^{1/2}$ -simple process. By linearity, we can suppose $X_s = x1_{[t_1, t_2)}(s)$, with $x \in D(\Delta^{1/2}) \cap L^2(M_{t_1})$. In that case, we have clearly $\eta_\alpha(X_s) \in D(\tilde{\delta} \circ \Delta)$ (using assumption d') and the equality stated is nothing but the one of lemma 2. In the general case $X_s \in \mathcal{B}_{2, \Delta^{1/2}}^a([0, T])$, take by density X_s^n $\Delta^{1/2}$ -simple processes converging to X_s in $\mathcal{B}_{2, \Delta^{1/2}}^a([0, T])$. Then, since $\delta \circ \Delta^\beta \eta_\alpha(X_s^n) = \delta \circ \alpha^\beta (id - \eta_\alpha)^\beta \eta_\alpha^{1-\beta}(X_s^n)$ (and using assumption c), $\sum_{b=0}^2 \|\delta \circ \Delta^{b/2} \eta_\alpha(X_s^n - X_s^m)\|_2^2 \leq (1 + 2\alpha + (2\alpha)^2) (\|\Delta^{1/2}(X_s^n - X_s^m)\|_2^2 + \omega \|X_s^n - X_s^m\|_2^2)$. Likewise

$$\sum_{b=0}^2 \|\Delta^{b/2} \eta_\alpha(X_s^n - X_s^m)\|_2^2 \leq (1 + 2\alpha + (2\alpha)^2) \|X_s^n - X_s^m\|_2^2.$$

As a consequence, $(\eta_\alpha(X_s^n))$ converges in $\mathcal{B}_{2, \delta \circ \Delta}^a([0, T])$ (and by the embedding in $\mathcal{B}_{2, \Delta^{1/2}}^a([0, T])$) it converges to $\eta_\alpha(X_s)$ which is thus in $\mathcal{B}_{2, \delta \circ \Delta}^a$. Now, $\Delta^\otimes \eta_\alpha^\otimes \tilde{\delta}(X_s), \tilde{\delta}(\Delta \eta_\alpha X_s) \in \mathcal{B}_2^a([0, T])$, therefore applying the equation of lemma 2, $H_\alpha(X_s \oplus \Delta^{1/2}(X_s)) \in \mathcal{B}_2^a([0, T])$ and we have our equality after taking the isometry of stochastic integration. The second statement of (i) is proved in a similar way using d' for the equation, g, h, e for boundedness results. For (ii), the boundedness has been already discussed, and this explains the definition of the right hand sides of the equations. The equalities are clear for simple processes (easy consequences of assumptions d (and a for the semigroup)), this concludes by density.

For the last statement of (ii) about orthogonality, since $\delta \eta_\alpha(V) \rightarrow \delta(V)$, we can assume by the beginning of (ii) (putting η_α in the stochastic integral), $U_s \in D(\Delta^\otimes)$. Again, it suffices to prove the simple process case, and this reduces to the last assumption in hypothesis d.

Finally, for the equivalence of (iii), note that the two first equalities of (iii) gives $\|\Delta^{1/2} \eta_\alpha(\int_0^T U_s \# dS_s)\|_2^2 = \int_0^T \|\Delta^{\otimes 1/2} \eta_\alpha^\otimes(U_s)\|_2^2$. From this, letting $\alpha \rightarrow \infty$, the direct implication follows from monotone convergence theorem with $\Delta^{1/2} \eta_\alpha = \eta_\alpha \Delta^{1/2}$ and the reverse implication using also $\Delta^{1/2}$ is a closed operator. Let us check first for any $U \in \mathcal{B}_2^a([0, T])$, $\Delta^{1/2} \eta_\alpha(\int_0^T U_s \# dS_s) = \int_0^T \Delta^{\otimes 1/2} \eta_\alpha^\otimes U_s \# dS_s$. Again it suffices to check it on simple processes, on which this comes from $\Delta^{1/2} = \int_0^\infty \pi^{-1} t^{-1/2} (id - \eta_t) dt$ (cf. e.g. [43] or [21]). Now, the stated result comes from $\alpha \rightarrow \infty$, the statement for $\tilde{\delta}$ is analogous. \square

1.2. A definition of free Stochastic convolution. In this subsection, we assume $\Gamma_0(\omega)$. We want to give sense to the following kind of integral, for $U_s \in \mathcal{B}_2^a : \int_0^t \phi_{t-s}(U_s \# dS_s)$. We will define it by

$$\int_0^t \phi_{t-s}(U_s \# dS_s) = \int_0^t \phi_{t-s}^\otimes(U_s) \# dS_s,$$

and we want to verify the usual properties of stochastic convolution.

For this, we have to verify that $\phi_{t-s}^\otimes(U_s)1_{[0, t]}(s) \in \mathcal{B}_2^a([0, t])$, and since ϕ^\otimes is a contraction, it is sufficient to show this for U_s a simple process, and thus even for $U1_{[u, v)}(s)$, $U \in L^2(M \otimes M)$. But consider $u_{i,n} = u + i(v - u)/n$, then $U^n = \sum_{i=0}^{n-1} \phi_{t-u_{i,n}}(U)1_{[u_{i,n}, u_{i+1,n})}$ is easily shown to converge in $L^2([0, t], L^2(M \otimes M))$ to $\phi_{t-s}^\otimes(U)$ using strong continuity of ϕ^\otimes , this concludes the preliminaries for the definition.

Let us define a variant of the spaces of the previous part useful to define a really weak form of solutions we will call in the next part “ultramild” solutions. We will write $\mathcal{B}_{2, \phi \delta}^a([0, T])$ for

the completion with respect to the following norm of δ -simple adapted processes, i.e. recall this means processes of the form $X = \sum_{j=1}^M X_j 1_{[t_j, t_{j+1})}$ with $X_j \in D(\delta)$:

$$\|X\|_{\mathcal{B}_{2,\phi\delta}^a} = \left(\int_0^T \int_0^t \|\phi_{t-s}^\otimes \delta(X_s)\|_{L^2(\tau \otimes \tau) \oplus \mathbb{N}}^2 ds + \|X_t\|_{L^2(\tau)}^2 dt \right)^{1/2}.$$

We have clearly a continuous embedding $\mathcal{B}_{2,\delta}^a([0, T]) \rightarrow \mathcal{B}_{2,\phi\delta}^a([0, T])$ using subsection 1.1 and the above remark defining stochastic convolution (the first space being clearly dense in the second by definition). We have thus a map $\gamma : \mathcal{B}_{2,\delta}^a([0, T]) \rightarrow L_a^2([0, T], L^2(M))$ such that $\gamma(X)_t = \int_0^t \phi_{t-s}(\delta(X_s)) \# dS_s$ and clearly $\|\gamma(X)\|_{L_a^2([0, T], L^2(M))} \leq \|X\|_{\mathcal{B}_{2,\phi\delta}^a}$ so that γ extends to a continuous map (also called) $\gamma : \mathcal{B}_{2,\phi\delta}^a([0, T]) \rightarrow L_a^2([0, T], L^2(M))$.

We also want to show that $t \mapsto \langle \int_0^t U_s \# dS_s, \zeta \rangle$ is of bounded variation so that we can remark that we can define something like $\int_0^t \langle U_s \# dS_s, \zeta \rangle$ (with the same value) and see $\langle U_s \# dS_s, \zeta \rangle$ as a measure on \mathbb{R}_+ . But since stochastic integration is an isometry onto its image we can project ζ on this space thus write its projection $\int_0^t V_s \# dS_s$, and the result is a consequence of the isometry property.

Finally, we want to define for $\zeta(\cdot) \in C^1([0, T], L^2(M))$: $\int_0^t \langle U_s \# dS_s, \zeta(s) \rangle$ and show a relation with stochastic convolution in a special case. For this, first note that the family of functions of the form $\varphi(\cdot)\zeta_0$, for $\varphi(\cdot) \in C^1([0, T], \mathbb{C})$ and $\zeta_0 \in L^2(M)$ linearly spans a dense subset of $C^1([0, T], L^2(M))$, thus consider also first $\zeta(\cdot)$ in this linear span. Consider also $\mathcal{U}_t = \int_0^t U_s \# dS_s$.

Define $\int_0^t \langle U_s \# dS_s, \varphi(s)\zeta_0 \rangle = \langle \int_0^t \overline{\varphi(s)} U_s \# dS_s, \zeta_0 \rangle$ using the previous paragraph, and consider like in this paragraph the projection of ζ_0 on the space of stochastic integrals $\int_0^t V_s \# dS_s$. Then compute using integration by parts :

$$\begin{aligned} \int_0^t \langle U_s \# dS_s, \varphi(s)\zeta_0 \rangle &= \int_0^t \varphi(s) \langle U_s, V_s \rangle ds \\ &= \varphi(t) \int_0^t \langle U_s, V_s \rangle ds - \int_0^t \varphi'(s) \langle \mathcal{U}_s, \zeta_0 \rangle ds \\ &= \langle \mathcal{U}_t, \zeta(t) \rangle - \int_0^t \langle \mathcal{U}_s, \zeta'(s) \rangle ds, \end{aligned}$$

which extends by linearity on the above mentioned linear span.

But now, we get the bound :

$$\begin{aligned} \left| \int_0^t \langle U_s \# dS_s, \varphi(s)\zeta_0 \rangle \right| &\leq \|\mathcal{U}_t\|_2 \left(\|\zeta(t)\|_2 + t \sup_s \|\zeta'(s)\|_2 \right) \\ &\leq \|\mathcal{U}_t\|_2 \max(1, t) \left(\sup_s \|\zeta(s)\|_2 + \sup_s \|\zeta'(s)\|_2 \right), \end{aligned}$$

using $\|\mathcal{U}_t\|_2$ is increasing with t (by the isometry definition of stochastic integral). We can thus extend our linear map by continuity to $\zeta(\cdot) \in C^1([0, T], L^2(M))$ and we have also the equality :

$$(3) \quad \int_0^t \langle U_s \# dS_s, \zeta(s) \rangle = \langle \mathcal{U}_t, \zeta(t) \rangle - \int_0^t \langle \mathcal{U}_s, \zeta'(s) \rangle ds.$$

Consider finally $\zeta(s) = \phi_{t-s}(\zeta)$, with $\zeta \in D(\Delta)$, writing as before $\int_0^\infty V_s \# dS_s$ the projection of ζ on the space of stochastic integrals. Using this last equality, we get :

$$\begin{aligned}
\int_0^t \langle U_s \# dS_s, \phi_{t-s}(\zeta) \rangle &= \langle \mathcal{U}_t, \zeta \rangle - \int_0^t \langle \frac{1}{2} \Delta \phi_{t-s}(\mathcal{U}_s), \zeta \rangle ds \\
&= \int_0^t \langle U_s, V_s \rangle ds - \int_0^t \int_0^s \langle \frac{1}{2} \Delta^\otimes \phi_{t-s}^\otimes(U_u), V_u \rangle du ds \\
&= \int_0^t \langle U_s, V_s \rangle ds - \int_0^t \langle \int_u^t \frac{1}{2} \Delta^\otimes \phi_{t-s}^\otimes(U_u) ds, V_u \rangle du \\
&= \int_0^t \langle U_s, V_s \rangle ds - \int_0^t \langle U_u - \phi_{t-u}^\otimes(U_u), V_u \rangle du \\
&= \langle \int_0^t \phi_{t-s}(U_s \# dS_s), \zeta \rangle
\end{aligned}$$

In the first line we used our identity since $\phi_{T-s}(\zeta) \in C^1([0, T], L^2(M))$ since $\zeta \in D(\Delta)$ (used to get differentiability at T).

We also used lemma 3 (ii) to get line 2.

Line 4 and 5 are only computations, first with the differential equation, second, with the definition of stochastic convolution after simplification.

In line 3, we have to justify application of Fubini theorem. Note that $\zeta = \eta_\alpha(z)$ (since by Hille-Yosida theory $\text{Range}(\eta_\alpha) = D(\Delta)$, see e.g. (1.3) in the proof of Chapter 1 proposition 1.5 in [28]), if the projection of z is written $\int_0^T W_s \# dS_s$, then $V_s = \eta_\alpha^\otimes(W_s)$ a.e. by lemma 3 (ii). Thus V_s is a.e. in $D(\Delta^\otimes)$. We can now use Cauchy-Schwarz inequality and Fubini-Tonelli Theorem :

$$\begin{aligned}
\int_0^t ds \int_0^s du |\langle \frac{1}{2} \Delta^\otimes \phi_{t-s}^\otimes(U_u), V_u \rangle| &\leq \int_0^t ds \left(\int_0^s du \|\frac{1}{2} \phi_{t-s}^\otimes(U_u)\|_2^2 \right)^{1/2} \left(\int_0^s du \|\Delta^\otimes V_u\|_2^2 \right)^{1/2} \\
&\leq t \left(\int_0^t du \|U_u\|_2^2 \right)^{1/2} \|\Delta(\zeta)\|_2 < \infty.
\end{aligned}$$

Starting from the third line above, applying Fubini to go upwards after a change of variable, we also obtain :

$$\begin{aligned}
\langle \int_0^t \phi_{t-s}(U_s \# dS_s), \zeta \rangle &= \int_0^t \langle U_s, V_s \rangle ds - \int_0^t \langle \int_u^t \frac{1}{2} \Delta^\otimes \phi_{t-s}^\otimes(U_u) ds, V_u \rangle du \\
&= \langle \mathcal{U}_t, \zeta \rangle - \int_0^t \langle \frac{1}{2} \Delta \int_0^s \phi_{s-u}(U_u \# dS_u), \zeta \rangle ds.
\end{aligned}$$

Proposition 4. (*Integration by parts for stochastic convolution*) For $\zeta \in D(\Delta)$, $U \in B_2^a([0, t])$, we have :

$$\begin{aligned}
\int_0^t \phi_{t-s}(U_s \# dS_s) &= \int_0^t U_s \# dS_s - \Delta \int_0^t ds \frac{1}{2} \phi_{t-s} \left(\int_0^s U_v \# dS_v \right) \\
&= \int_0^t U_s \# dS_s - \frac{1}{2} \Delta \int_0^t ds \int_0^s \phi_{s-u}(U_u \# dS_u)
\end{aligned}$$

$$\int_0^t \langle U_s \# dS_s, \phi_{t-s}(\zeta) \rangle = \langle \int_0^t \phi_{t-s}(U_s \# dS_s), \zeta \rangle$$

Note the following useful formula we will use often later :

$$(4) \quad \|\phi_t(x)\|_2^2 = \|x\|_2^2 - \int_0^t \|\Delta^{1/2} \phi_s(x)\|_2^2 ds.$$

Proposition 5. For $Y \in \mathcal{B}_{2,\delta}^a([0, T])$, define $\gamma(Y)_t = \int_0^t \phi_{t-s}(\delta(Y_s)) \# dS_s$, then $\gamma(Y)_t \in D(\Delta^{1/2})$ for a.e. $t \leq T$ and moreover :

$$(5) \quad \int_0^T \|\Delta^{1/2} \gamma(Y)_t\|_2^2 dt = -\|\gamma(Y)_T\|_2^2 + \int_0^T dt \|\delta(Y_t)\|_2^2$$

Moreover, assume $\Gamma_1(\omega, C)$ for any B among $\tilde{\delta}, \Delta^{1/2}$, $\alpha, \alpha' > 0$, then :

$$(6) \quad \|B\eta_\alpha^{1/2}\eta_{\alpha'}^{1/2}(\gamma(Y)_T)\|_2^2 = \|B\eta_\alpha^{1/2}\eta_{\alpha'}^{1/2} \int_0^T \delta(Y_s) \# dS_s\|_2^2 - \int_0^T dt \Re \langle B\Delta\eta_\alpha^{1/2}\eta_{\alpha'}^{1/2}\gamma(Y)_t, B\eta_\alpha^{1/2}\eta_{\alpha'}^{1/2}\gamma(Y)_t \rangle$$

Proof. By Fubini-Tonelli Theorem and the remark before the proposition, we deduce

$$\int_0^T dt \int_0^t ds \|\Delta^{1/2} \phi_{t-s}(\delta(Y_s))\|_2^2 = \int_0^T ds \|\delta(Y_s)\|_2^2 - \int_0^T dt \|\phi_{T-t}(\delta(Y_t))\|_2^2.$$

Thus lemma 3 (iii) concludes the first statement. Since $B\eta_\alpha$ is a bounded operator, and from the first statement and $\Gamma_1(\omega, C)$ c) for the last term, all the terms in (6) are continuous on $\mathcal{B}_{2,\delta}^a([0, T])$. As a consequence, it suffices to prove it for simple processes of even $Y = X1_{[s,t]}$, $X \in D(\delta)$.

From lemma 3 (and with an obvious notation B^\otimes), this reduces the statement to

$$\begin{aligned} & \int_s^t du \|B^\otimes \eta_\alpha^{\otimes 1/2} \eta_{\alpha'}^{\otimes 1/2} \phi_{T-u}^\otimes \delta(X)\|_2^2 = \|B^\otimes \eta_\alpha^{\otimes 1/2} \eta_{\alpha'}^{\otimes 1/2} (\delta(X))\|_2^2 (t-s) \\ & - \int_s^t dv \Re \int_s^v du \langle B^\otimes \Delta^\otimes \eta_\alpha^{\otimes 1/2} \eta_{\alpha'}^{\otimes 1/2} \phi_{v-u}^\otimes \delta(X), B^\otimes \eta_\alpha^{\otimes 1/2} \eta_{\alpha'}^{\otimes 1/2} \phi_{v-u}^\otimes \delta(X) \rangle \end{aligned}$$

But this is obvious after applying Fubini on the last integral and integrating along v . \square

1.3. Useful links between mild solutions and strong solutions. In this part, we will also work under assumption $\Gamma_0(\omega)$. Let us define two kinds of solutions.

Definition 6. We will call a **strong solution** an element $X_t \in \mathcal{B}_{2,\Delta}^a$ satisfying (2). A **mild solution** will be an $X_t \in \mathcal{B}_{2,\delta}^a$ satisfying :

$$(7) \quad X_t = \phi_t(X_0) + \int_0^t \phi_{t-s}(\delta(X_s)) \# dS_s.$$

We call **ultramild solutions**, solutions of (7) in $\mathcal{B}_{2,\phi\delta}^a$. We call a **weak solution** an $X_t \in \mathcal{B}_{2,\delta}^a$ such that, for any $\zeta \in D(\Delta)$:

$$\langle X_t, \zeta \rangle = \langle X_0, \zeta \rangle - \frac{1}{2} \int_0^t \langle X_s, \Delta(\zeta) \rangle ds + \int_0^t \langle \delta(X_s) \# dS_s, \zeta \rangle.$$

We will first recall analogs of usual results (in classical SPDE theory) concerning the link between strong solutions and mild solutions. We mainly follow here the proofs (for a classical Brownian motion and a classical SPDE) of [13] Chapter 6.

Proposition 7. *A solution of (2) (in $\mathcal{B}_{2,\Delta}^a$) is also a mild solution (even a solution of (7) in $\mathcal{B}_{2,\Delta^{1/2}}^a$.)*

Proof. First, note that for any $\zeta(\cdot) \in C^1([0, T]; D(\Delta))$, and any $t \in [0, T]$, we have :

$$\langle X_t, \zeta(t) \rangle = \langle X_0, \zeta(0) \rangle + \int_0^t \langle X_s, -\frac{1}{2} \Delta(\zeta(s)) + \zeta'(s) \rangle ds + \int_0^t \langle \delta(X_s) \# dS_s, \zeta(s) \rangle.$$

(To prove this, use (3) to compute the stochastic part and use an integration by parts to get the other term).

Finally, consider $\zeta(s) = \phi_{t-s}(\zeta)$ (which is in $C^1([0, T]; D(\Delta))$ say $\zeta \in D(\Delta^2)$ by the differential equation that ϕ satisfies). The terms inside the usual integral cancel out and you get :

$$\langle X_t, \zeta \rangle = \langle X_0, \phi_t(\zeta) \rangle + \int_0^t \langle \delta(X_s) \# dS_s, \phi_{t-s}(\zeta) \rangle.$$

Inasmuch as you can take any $\zeta \in D(\Delta^2)$ and $D(\Delta^2)$ is dense (even a core for Δ by a standard result Theorem 3.24 p275 in Chapter V of Kato's book [21]), you get the result (using proposition 4). \square

Proposition 8. *A mild solution X_t is always a weak solution, and if it is also in $\mathcal{B}_{2,\Delta}^a$, then it is in fact a strong solution.*

Proof. Once we have proved that our mild solution is in fact a weak solution, we are in fact done, since on our assumption $\Delta(X_s)$ is (lebesgue-almost surely) well defined and in $L^2([0, T], L^2(M))$, showing that the wanted equation (2) under $\langle \cdot, \zeta \rangle$ which concludes by density.

To show that we have the desired weak solution, we will merely use that the solution is in $\mathcal{B}_{2,\Delta^{1/2}}^a$. Consider thus $\zeta \in D(\Delta^2)$

$$\begin{aligned}
& -\frac{1}{2} \int_0^t \langle X_s, \Delta(\zeta) \rangle ds \\
&= -\frac{1}{2} \int_0^t \langle X_0, \phi_s(\Delta(\zeta)) \rangle ds - \frac{1}{2} \int_0^t ds \int_0^s \langle \delta(X_u) \# dS_u, \phi_{s-u}(\Delta(\zeta)) \rangle \\
&= -\frac{1}{2} \langle X_0, \int_0^t \phi_s(\Delta(\zeta)) ds \rangle + \left(\left\langle \int_0^t \phi_{t-s}(\delta(X_s) \# dS_s), \zeta \right\rangle - \left\langle \int_0^t \delta(X_s) \# dS_s, \zeta \right\rangle \right) \\
&= \langle \phi_t(X_0) + \int_0^t \phi_{t-s}(\delta(X_s) \# dS_s), \zeta \rangle - \langle X_0, \zeta \rangle - \int_0^t \langle \delta(X_s) \# dS_s, \zeta \rangle \\
&= \langle X_t, \zeta \rangle - \langle X_0, \zeta \rangle - \int_0^t \langle \delta(X_s) \# dS_s, \zeta \rangle
\end{aligned}$$

The first line has been justified in proposition 4 applied to the definition of mild solutions. The last line, clearly concluding to what we wanted to prove, uses nothing but the definition of a mild solution, also used in the first line. Of course the third line uses again the differential equation for ϕ . The second line reduces to the second equation in proposition 4. \square

Finally, to get uniqueness results even in the case of really weak conditions, we want to introduce a notion of ultraweak solution for which uniqueness will be easy to prove so that we will build unique ultraweak solutions which are also ultramild solutions. This needs some results on chaotic decomposition very similar to those of section 5.3 in [2] but not only for the free Fock space $F(H)$ with $H = L^2(\mathbb{R}_+)$ but also for $H = L^2(\mathbb{R}_+)^{\oplus \mathbb{N}}$ and moreover with an initial condition space $L^2(M_0)$ i.e. we want to see a multiple stochastic integral variant of $L^2(M) = L^2(M_0 \star \mathcal{SC}(H)) \simeq L^2(M_0) \star F(H)$. Since this requires a little bit of notation with nothing new, we merely state the results after introduction of notations.

For $f \in L^2(\mathbb{R}_+^n \times \mathbb{N}^n, L^2(M_0)^{n+1})$, we want to define a stochastic integral

$I(f) = \int f(t_1, \dots, t_n) \# dS_{t_1} \dots dS_{t_n}$. Of course, we extend it by isometry, and linearity like in [2] after defining it on appropriate multiple of characteristic function $f = 1_{\mathcal{A}} \delta_{k_1, \dots, k_n} \alpha_0 \otimes \dots \otimes \alpha_n$, $\alpha_i \in L^2(M_0)$, δ_{k_1, \dots, k_n} the function on \mathbb{N}^n taking non zero value 1 only on the indicated support, $\mathcal{A} = [u_1, v_1] \times \dots \times [u_n, v_n]$ with $\mathcal{A} \subset \mathbb{R}_+^n - D^n$ (D^n the usual diagonal e.g. definition 5.3.1 of [2]) by :

$$I(f) := \alpha_0 (S_{v_1}^{(k_1)} - S_{u_1}^{(k_1)}) \dots (S_{v_n}^{(k_n)} - S_{u_1}^{(k_n)}) \alpha_n.$$

Then we can write $f = \sum_{n=0}^{\infty} f_n \in L^2(M_0) \star F(H)$ so that $I(f) = \sum_{i=0}^{\infty} I(f_n)$ define an isometry $I : L^2(M_0) \star F(H) \rightarrow L^2(M)$ determined by $I(f)\Omega = f$ (Ω the usual cyclic empty vector in Fock space), like in proposition 5.3.2 of [2]. Recall P_{Γ} is the projection on adapted bi-processes. It is defined (as Γ) in proposition 5.3.12 in [2] before free Bismut-Clark-Ocone formula (also valid in our context, recall it involves ∇_s the gradient operator from definition 5.1.1 in [2]) .

We can now define :

Definition 9. An **ultraweak solution** (of (2)) is a weakly continuous adapted process X_t in $L_{a,loc}^2(\mathbb{R}_+, L^2(M))$ such that, for some C and ω , $\|X_t\|_2 \leq Ce^{\omega t}$ and for all finite sums

$g = \sum_n g_n$, $g_n \in L^2(\mathbb{R}_+^n \times \mathbb{N}^n) \otimes_{alg} D(\Delta)^{\otimes n+1}$ step function as above, then a.e. in $t \in \mathbb{R}_+$:

$$\langle I(g), X_t \rangle = \langle I(g), X_0 \rangle - \frac{1}{2} \int_0^t ds \langle \Delta I(g), X_s \rangle + \int_0^t ds \langle \delta^* P_\Gamma \nabla_s I(g), X_s \rangle.$$

1.4. Mild and Ultramild solutions. Here is the main theorem in the general setting.

Theorem 10. (i) *Let us assume $\Gamma_0(\omega)$ and that $X_0 \in L^2(M_0)$, then equation (2) has a unique ultraweak solution. This solution is also an Ultramild Solution X_t and we have, for every T and a.e. in t :*

$$\|X_t\|_2^2 \leq e^{\omega t} \|X_0\|_2^2$$

$$\|X\|_{\mathcal{B}_{2,\phi\delta}^a([0,T])}^2 \leq \|X_0\|_2^2 2 \frac{e^{\omega T} - 1}{\omega} \quad (\text{or } 2T\|X_0\|_2^2 \text{ if } \omega = 0).$$

Furthermore, if we write X_s^ϵ a solution for δ replaced by $(1-\epsilon)\delta$ ($\epsilon \in (0, 1]$), then X_s^ϵ is a unique mild solution of this variant equation, i.e. a solution of (7 ϵ) in $\mathcal{B}_{2,\Delta^{1/2}}^a$, and the solution built above X_t is, for every T , a weak limit ($\epsilon \rightarrow 0$) in $\mathcal{B}_{2,\phi\delta}^a([0, T])$ and strong limit in $C^0([0, T], (L^2(M), \sigma(L^2(M), L^2(M))))$ of the solutions X_t^ϵ . Finally, if we assume $X_0 \in D(\Delta^{\otimes 1/2}\delta) \cap D(\Delta) \cap L^2(M_0)$ then the solution satisfies a.e.:

$$\begin{aligned} \|X_t - X_0 - \delta(X_0) \# S_t\|_2^2 &\leq \frac{t^2}{4} \|\Delta(X_0)\|_2^2 + (e^{\omega t} - 1) \|X_0\|_2^2 \\ &+ \frac{t^2}{2} \left(\|\Delta^{\otimes 1/2}(\delta(X_0))\|_2^2 + \frac{\pi}{4} (\|\Delta(X_0)\|_2^2 + \omega \|\Delta^{1/2}(X_0)\|_2^2)^{1/2} \|\Delta^{\otimes 1/2}(\delta(X_0))\|_2 \right) \\ &+ t \sup_{s \in [0, t]} (\|\Delta^{1/2}(\phi_s X_0)\|_2^2 - \|\delta(\phi_s X_0)\|_2^2) + t^2 \omega / 4 \|\Delta^{1/2}(X_0)\|_2^2. \end{aligned}$$

(ii) *Let us assume $\Gamma_1(\omega, C)$ and that $X_0 \in D(\Delta^{1/2}) \cap L^2(M_0)$, then equation (2) has a unique **Mild Solution** X_t . Moreover, we have the following inequalities a.e:*

$$\|X_t\|_2^2 \leq e^{\omega t} \|X_0\|_2^2$$

$$\|\tilde{\delta}(X_t)\|_2^2 \leq \|\tilde{\delta}(X_0)\|_2^2 e^{(6+2\omega)C^4 t}.$$

If we write X_s^ϵ a solution for δ replaced by $(1-\epsilon)\delta$ ($\epsilon \in (0, 1]$), then, if $\delta = \tilde{\delta}$, X_s^ϵ is a strong solution, i.e. a solution of (7 ϵ) in $\mathcal{B}_{2,\Delta}^a([0, T])$, for every T , and otherwise, if $\delta \neq \tilde{\delta}$ a mild solution by (i). Furthermore, X_t is, for every T , the weak limit ($\epsilon \rightarrow 0$) in $\mathcal{B}_{2,\delta}^a([0, T])$ and strong limit in $\mathcal{B}([0, T], L^2(M))$ (the space of bounded functions with uniform convergence) of the solution X_t^ϵ .

Proof. Let us sketch the plan of the proof. Step 0 proves uniqueness of ultraweak solutions, which is a useful preliminary. We will first find unique mild (resp strong in case (ii)) solutions after replacing δ by $(1-\epsilon)\delta$ with $\epsilon > 0$ [step 1]. Then, we will prove that when $\epsilon \rightarrow 0$ we can get some weak convergence to an ultramild (resp a mild, in case (ii)) solution of (7), mainly by showing several inequalities like the ones stated in the theorem [step 2 for part (i), step 3 for part (ii)]

Step 0: Unicity of ultraweak solutions in case (i).

We have to show that an ultraweak solution with $X_0 = 0$ vanishes. The proof is in the spirit of Theorem 5.6 in [17] in the symmetric Fock space context. For g_n like in definition 9, we prove by induction on n $\langle I(g_n), X_s \rangle = 0$. The induction hypothesis or only the definition of ∇_s at initialization, gives the last integral in the definition of ultraweak solution vanishes, so that for $g = g_n$:

$$\langle I(g), X_t \rangle = -\frac{1}{2} \int_0^t ds \langle \Delta I(g), X_s \rangle.$$

Since $\|X_t\|_2 \leq C \exp(\omega t)$, we can consider the Laplace transform for $\lambda > \omega$ so that we get :

$$\begin{aligned} \lambda \int_0^\infty dt \exp(-\lambda t) \langle I(g), X_t \rangle &= -\frac{\lambda}{2} \int_0^\infty dt \exp(-\lambda t) \int_0^t ds \langle \Delta I(g), X_s \rangle \\ &= -\frac{\lambda}{2} \int_0^\infty ds \langle \Delta I(g), X_s \rangle \int_s^\infty dt \exp(-\lambda t) \\ &= -\frac{1}{2} \int_0^\infty dt \exp(-\lambda t) \langle \Delta I(g), X_t \rangle \end{aligned}$$

Thus $\langle (\lambda + \Delta/2)I(g), X_t \rangle = 0$ but $I((\lambda + \Delta^{\otimes(n+1)}/2)^{-1}(g)) = (\lambda + \Delta/2)^{-1}I(g)$ thus a change of g gives the result of the next inductive step. Now by density of the functions of the form $I(\sum g_n)$ (as in definition of ultraweak solutions) we get $X_t = 0$.

Step 1: Assume $\Gamma_0(\omega)$. For any $\epsilon \in (0, 1]$ and $X_0 = X_0^\epsilon \in L^2(M_0)$ there exists a unique mild solution (even in $\mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$ for any T) to $X_t^\epsilon = \phi_t(X_0) + (1 - \epsilon)\gamma(X^\epsilon)_t$

Assume now $\Gamma_1(\omega, C)$ and $\delta = \tilde{\delta}$. For any $\epsilon \in (0, 1]$ and $X_0 = X_0^\epsilon \in D(\Delta^{1/2}) \cap L^2(M_0)$ there exists a unique strong solution (i.e. in $\mathcal{B}_{2,\Delta}^a([0, T])$ for any T) to $X_t^\epsilon = \phi_t(X_0) + (1 - \epsilon)\gamma(X^\epsilon)_t$

For each statement we can be content with proving for a small $T > 0$ to be fixed later. Then, using the fact that Γ_0, Γ_1 are translation invariant, if we consider the same problem starting at kT , this gives the same thing on any $[0, T]$.

The first statement is easy and a consequence of (5) in Proposition 5. If $Y \in \mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$, define an element at least in $L^2(M_t) \cap D(\Delta^{1/2})$ (for a.e. $t \in (0, T]$, by Proposition 5 and since $\mathcal{B}_{2,\Delta^{1/2}}^a([0, T]) \hookrightarrow \mathcal{B}_{2,\delta}^a([0, T])$) :

$$\Gamma(Y)_t = \phi_t(X_0) + (1 - \epsilon) \int_0^t \phi_{t-s}(\delta(Y_s) \# dS_s).$$

First of all, $\Gamma(Y)$ is in $\mathcal{B}_{2,\Delta^{1/2}}^a$ for Y in this space. Indeed, first $\phi_t(X_0)$ is in this space, as a limit (coming from (4)) of $\phi_t(\eta_\alpha(X_0))$, continuous function in $C^0([0, T], D(\Delta^{1/2})) \hookrightarrow \mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$ (a usual $\Delta^{1/2}$ -simple-process approximation giving this). Second, since, if Y_n is a $\Delta^{1/2}$ -simple process converging to Y , $\gamma(Y_n)$ converge to $\gamma(Y)$ (a priori in $L^2([0, T], D(\Delta^{1/2}))$ from Proposition 5 (5)), it suffices to note $\gamma(Y_n)$ is itself in $\mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$.

Finally it suffices to check Γ is a contraction (after moving to an equivalent norm) on $\mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$. Indeed note from proposition 5 and the definition:

$$(8) \quad \begin{aligned} \int_0^T ds \|\Delta^{1/2}(\Gamma(Y)_s - \Gamma(Z)_s)\|_2^2 &\leq (1 - \epsilon)^2 \int_0^T ds \|\delta(Y - Z)_s\|_2^2 \\ \int_0^T ds \|\Gamma(Y)_s - \Gamma(Z)_s\|_2^2 &\leq (1 - \epsilon)^2 T \int_0^T ds \|\delta(Y - Z)_s\|_2^2. \end{aligned}$$

Thus fix $0 < T < \epsilon/(2 \max(1, \omega))$ so that one can take $K = \epsilon/2T$ to get $(1 - \epsilon + KT) \max(1, \omega) < (1 - \epsilon/2)K$ and define the equivalent norm $\|Y\|_{\mathcal{B}_{2,\Delta^{1/2}}^a, K} = \int_0^T ds \|\Delta^{1/2}(Y)_s\|_2^2 + K\|Y_s\|_2^2$ on $\mathcal{B}_{2,\Delta^{1/2}}^a$, we deduce from $\Gamma_0(\omega)$ c) and the previous inequalities that

$$\begin{aligned} \|\Gamma(Y) - \Gamma(Z)\|_{\mathcal{B}_{2,\Delta^{1/2}}^a, K} &\leq ((1 - \epsilon) + KT) \left(\int_0^T ds \|\Delta^{1/2}(Y - Z)_s\|_2^2 + \max(1, \omega) \|(Y - Z)_s\|_2^2 \right) \\ &\leq (1 - \epsilon/2) \|Y - Z\|_{\mathcal{B}_{2,\Delta^{1/2}}^a, K}. \end{aligned}$$

This concludes to the first statement.

For the second statement, we want to show Γ is a contraction on $\mathcal{B}_{2,\Delta}^a([0, T])$ after taking an equivalent norm again. We thus now take $Y \in \mathcal{B}_{2,\Delta}^a([0, T])$.

We can apply Proposition 5 (6) to get :

$$\begin{aligned} \int_0^T dt \|\Delta \eta_\alpha^{1/2} \eta_{\alpha'}^{1/2} \gamma(Y)_t\|_2^2 &= -\|\Delta^{1/2} \eta_\alpha^{1/2} \eta_{\alpha'}^{1/2} (\gamma(Y)_T)\|_2^2 + \int_0^T dt \|\Delta^{\otimes 1/2} \eta_\alpha^{\otimes 1/2} \eta_{\alpha'}^{\otimes 1/2} (\delta(Y_t))\|_2^2 \\ &\leq \langle \Delta \eta_\alpha \int_0^T \delta(Y_t) \# dS_t, \int_0^T \delta(Y_t) \# dS_t \rangle, \end{aligned}$$

where we have used in the second line lemma 3 (ii) and contractivity of $\eta_{\alpha'}^{1/2}$. But now (in the case we assume Γ_1 and $\delta = \tilde{\delta}$), we can use lemma 3 (i) and then Γ_1 c') and the bound in lemma 2 for \mathcal{H}_α to get :

$$\begin{aligned} &\langle \Delta \eta_\alpha \int_0^T \delta(Y_t) \# dS_t, \int_0^T \delta(Y_t) \# dS_t \rangle \\ &= \langle \int_0^T \delta \circ \Delta(\eta_\alpha(Y_s)) \# dS_s + \int_0^T \mathcal{H}_\alpha(Y_s \oplus \Delta^{1/2}(Y_s)) \# dS_s, \int_0^T \delta(Y_t) \# dS_t \rangle \\ &\leq \int_0^T \|\Delta(Y_t)\|_2^2 dt + (\max(1, \omega)C + \omega) \int_0^T \|Y_t\|_2^2 + \|\Delta^{1/2}(Y_t)\|_2^2 dt. \end{aligned}$$

But better, we can write $\|\Delta \eta_\alpha^{1/2} \eta_{\alpha'}^{1/2} \gamma(Y)_t\|_2^2 = \langle \Delta \eta_\alpha (\Delta)^{1/2} \eta_{\alpha'}^{1/2} \gamma(Y)_t, (\Delta)^{1/2} \eta_{\alpha'}^{1/2} \gamma(Y)_t \rangle$ to show that this increases to $\|\Delta \eta_{\alpha'}^{1/2} \gamma(Y)_t\|_2^2$ in α and then to $\|\Delta \gamma(Y)_t\|_2^2$ in α' , with the inequality below and as a consequence (recall $C \geq 1$) $\gamma(Y)_t \in D(\Delta)$ a.e. and we got :

$$(9) \quad \int_0^T dt \|\Delta \gamma(Y)_t\|_2^2 \leq \int_0^T \|\Delta(Y_t)\|_2^2 dt + 2 \max(1, \omega)C \int_0^T \|Y_t\|_2^2 + \|\Delta^{1/2}(Y_t)\|_2^2 dt.$$

Time has gone to choose T small enough and introduce the equivalent norm on $\mathcal{B}_{2,\Delta}^a([0, T])$ for which Γ will be a contraction under the assumption of (ii).

First choose T such that $T\omega < 1 - (1 - \epsilon)^2$ so that $T\omega < \frac{1}{(1-\epsilon)^2} - 1$. Second, let $L > \frac{2C \max(1, \omega)(1+T)}{1-(1-\epsilon)^2-\omega T} > 0$, and $K = L\omega + 2C \max(1, \omega) > 0$ thus :

$$L > L\eta := L(1-\epsilon)^2 + (2C \max(1, \omega)(1+T) + \omega TL)(1-\epsilon)^2 = L(1-\epsilon)^2 + (2C \max(1, \omega) + TK)(1-\epsilon)^2.$$

We get also :

$$K > K\eta' := (1 - \epsilon)^2 K(1 + T\omega) = (1 - \epsilon)^2 (L\omega + 2C \max(1, \omega) + KT\omega).$$

Finally define the clearly equivalent norm : $\|X\|_{L,K,T}^2 = \int_0^T L \|\Delta^{1/2}(X_s)\|_{L^2(\tau)}^2 + K \|X_s\|_{L^2(\tau)}^2 + \|\Delta(X_s)\|_{L^2(\tau)}^2 ds$. We get, using (8) and (9) in the first line, and then assumption c in the second line :

$$\begin{aligned} (1 - \epsilon)^2 \|\gamma(Y)\|_{L,K,T}^2 &\leq (1 - \epsilon)^2 \int_0^T \|\Delta(Y_t)\|_2^2 dt \\ &\quad + (1 - \epsilon)^2 2C \max(1, \omega) \int_0^T \|\Delta^{1/2}(Y_t)\|_2^2 + \|Y_t\|_2^2 dt + (L + KT)(1 - \epsilon)^2 \int_0^T \|\delta(Y_t)\|_2^2 dt \\ &\leq (1 - \epsilon)^2 \int_0^T \|\Delta(Y_t)\|_2^2 dt + L\eta \int_0^T \|\Delta^{1/2}(Y_t)\|_2^2 dt + K\eta' \int_0^T \|Y_t\|_2^2 dt \\ &\leq \max((1 - \epsilon)^2, \eta, \eta') \|Y\|_{L,K,T}^2. \end{aligned}$$

First of all, this shows that $\Gamma(Y)$ is indeed in $\mathcal{B}_{2,\Delta}^a$ for Y in this space, first since $\phi_t(X_0)$ is in this space as before and second, since, if Y_n is a Δ -simple process converging to Y , $\gamma(Y_n)$ converge to $\gamma(Y)$ (a priori in $L^2([0, T], D(\Delta))$, and $\gamma(Y_n)$ is itself in $\mathcal{B}_{2,\Delta}^a$).

Then, we can say that Γ is a contraction on $\mathcal{B}_{2,\Delta}^a([0, T])$ equipped of the norm $\|\cdot\|_{L,K,T}$, this concludes.

Step 2: Conclusion of the proof of (i).

Applying orthogonality (via lemma 3 (iii)) and equation (5) on $(1-\epsilon)\gamma(X^\epsilon)_t = X_t^\epsilon - \phi_t(X_0)$, we know that for any T :

$$(10) \quad \int_0^T dt \|\Delta^{1/2} X_t^\epsilon\|_2^2 = \int_0^T dt \|\Delta^{1/2} \phi_t(X_0)\|_2^2 - (1 - \epsilon)^2 \|(\gamma(X^\epsilon)_T)\|_2^2 + (1 - \epsilon)^2 \int_0^T dt \|(\delta(X_t^\epsilon))\|_2^2.$$

Using equation (4) and orthogonality and then assumption c we deduce :

$$\begin{aligned} (1 - \epsilon)^2 \|\gamma(X^\epsilon)_t\|_2^2 &\leq \|X_0\|_2^2 - \|\phi_t(X_0)\|_2^2 + \omega \int_0^t \|X_s^\epsilon\|_2^2 ds. \\ \|\gamma(X^\epsilon)_t\|_2^2 &= \|X_0\|_2^2 + (1 - \epsilon)^2 \left\| \int_0^t \delta(X_s^\epsilon) \# dS_s \right\|_2^2 - \int_0^t \|\Delta^{1/2}(X_s^\epsilon)\|_2^2 ds \leq \|X_0\|_2^2 + \omega \int_0^t \|X_s^\epsilon\|_2^2 ds. \end{aligned}$$

Note this second inequality works for $\epsilon = 0$ as soon as we have a solution in this case. We can use Gronwall's lemma (e.g. Th 0 in [58]) on this second inequality. It proves the first

inequality of the theorem (for X^ϵ as X). Combining this with the first inequality, we get after integration the second inequality in part (i), showing that X^ϵ is bounded in $\mathcal{B}_{2,\phi\delta}^a$.

$$\begin{aligned} \|X^\epsilon\|_{\mathcal{B}_{2,\phi\delta}^a} &= \left(\int_0^T \int_0^t \|\phi_{t-s}^\otimes \delta(X_s)\|_{L^2(\tau \otimes \tau) \oplus \mathbb{N}}^2 ds + \|X_t\|_{L^2(\tau)}^2 dt \right)^{1/2} \\ &\leq \frac{1}{(1-\epsilon)} \left(\int_0^T \left(\|X_0\|_{L^2(\tau)}^2 + \omega \int_0^t ds e^{\omega s} \|X_0\|_{L^2(\tau)}^2 \right) + e^{\omega t} \|X_0\|_{L^2(\tau)}^2 dt \right)^{1/2} \\ &\leq \frac{1}{(1-\epsilon)} \left(\int_0^T 2e^{\omega t} \|X_0\|_{L^2(\tau)}^2 dt \right)^{1/2}. \end{aligned}$$

Modulo extraction, we get a *-weak limit in $\mathcal{B}_{2,\phi\delta}^a([0, T])$ by compactity. As a consequence, since γ is a linear continuous map as recalled in the part on stochastic convolution, $\gamma(X^\epsilon)$ (or at least the image of the previous extraction) converges in $L^2([0, T], L^2(M))$ weakly. Since $\phi_t(X_0)$ is a constant in this space we can take the limit and verify the equation in this space, thus a.e., we especially get an ultramild solution. Since we deduce any such *-weak limit point is also an ultraweak solution (since X^ϵ is a mild thus weak thus ultraweak solution of the ϵ variant) we get *-weak convergence from uniqueness proved in step 0.

Moreover, taking $\xi \in L^2(M)$, with, say, the projection of ξ on the space of stochastic integrals given by $\int_0^T \eta_s \# dS_s$, let us prove that $\langle \xi, X_t^\epsilon \rangle$ is an equicontinuous and uniformly bounded family (for $\epsilon \in (0, 1]$) on $[0, T]$. From what we obtained above, only equicontinuity need to be proved, but (for $t \leq \tau$) we have (using the equation for X_τ^ϵ and Cauchy-Schwarz):

$$\begin{aligned} \langle \xi, X_\tau^\epsilon - X_t^\epsilon \rangle &\leq \|\xi\|_2 \|\phi_{\tau-t}(X_0) - X_0\|_2 \\ &\quad + (1-\epsilon) \int_t^\tau ds \langle \eta_s, \phi_{\tau-s}^\otimes(\delta(X_s^\epsilon)) \rangle + (1-\epsilon) \int_0^t ds \langle \phi_{\tau-t}^\otimes \eta_s - \eta_s, \phi_{t-s}^\otimes(\delta(X_s^\epsilon)) \rangle \\ &\leq \|\xi\|_2 \|\phi_{\tau-t}(X_0) - X_0\|_2 + \left(\int_t^\tau ds \|\eta_s\|_2^2 \right)^{1/2} \|(1-\epsilon)\gamma_\tau(X^\epsilon)\|_2 \\ &\quad + \left(\int_0^t \|\phi_{\tau-t}^\otimes \eta_s - \eta_s\|_2^2 \right)^{1/2} \|(1-\epsilon)\gamma_t(X^\epsilon)\|_2 \\ &\leq \|\xi\|_2 \|\phi_{\tau-t}(X_0) - X_0\|_2 + e^{\omega\tau/2} \|X_0\|_2 \left(\left(\int_t^\tau ds \|\eta_s\|_2^2 \right)^{1/2} + \|\phi_{\tau-t}\xi - \xi\|_2 \right). \end{aligned}$$

This concludes using strong continuity of ϕ_t (and using Heine-Cantor Theorem). As a consequence, using Arzela-Ascoli Theorem (and separability assumption on $L^2(M)$), we get via diagonal extraction, X_t is weakly continuous, and limit of a subsequence of X_t^ϵ in $C^0([0, T], (L^2(M), \sigma(L^2(M), L^2(M))))$. As a consequence, this easily enables us to pass to the limit $\epsilon \rightarrow 0$ in the first inequality of the theorem. From this we get also that any limit point is an ultraweak solution, so that from uniqueness we get the stated limit without extraction.

We now establish the supplementary inequality.

First by orthogonality and assumption d) of $\Gamma_0(\omega)$, we have :

$$\begin{aligned} & \|X_T^\epsilon - X_0 - (1 - \epsilon)\delta(X_0)\#S_T\|_2^2 \\ &= \|\phi_T(X_0) - X_0\|_2^2 + (1 - \epsilon)^2 \int_0^T dt \|\phi_{T-t}^\otimes \delta(\phi_t(X_0)) - \delta(X_0)\|_2^2 + (1 - \epsilon)^4 \|\gamma \circ \gamma(X^\epsilon)_T\|_2^2. \end{aligned}$$

Moreover, the same kind of orthogonality and relations (5) and (4) imply that :

$$\begin{aligned} (1 - \epsilon)^4 \|\gamma \circ \gamma(X^\epsilon)_T\|_2^2 &= \int_0^T dt ((1 - \epsilon)^2 \|\delta(X_t^\epsilon)\|_2^2 - \|\Delta^{1/2}(X_t^\epsilon)\|_2^2) \\ &+ \int_0^T dt (\|\Delta^{1/2}(\phi_t(X_0))\|_2^2 - (1 - \epsilon)^2 \|\delta(\phi_t(X_0))\|_2^2) + (1 - \epsilon)^2 \|\Delta^{1/2}(\gamma(\phi_t(X_0)))\|_2^2 \\ &\leq (e^{\omega T} - 1) \|X_0\|_2^2 + T \sup_{[0, T]} (\|\Delta^{1/2}(\phi_t X_0)\|_2^2 - (1 - \epsilon)^2 \|\delta(\phi_t X_0)\|_2^2) \\ &+ (1 - \epsilon)^2 \int_0^T dt \|\delta(\phi_t(X_0))\|_2^2 - \|\phi_{T-t} \delta(\phi_t(X_0))\|_2^2, \end{aligned}$$

where we used, in the first inequality, the first inequality of our theorem.

Our first line is in our estimate, it only remains to get the other terms by several elementary computations (only involving X_0).

$$\begin{aligned} & \int_0^T dt \|\phi_{T-t}^\otimes \delta(\phi_t(X_0)) - \delta(X_0)\|_2^2 + \|\delta(\phi_t(X_0))\|_2^2 - \|\phi_{T-t} \delta(\phi_t(X_0))\|_2^2 = \int_0^T dt \|\delta(\phi_t X_0 - X_0)\|_2^2 \\ &+ 2\Re \int_0^T dt \langle \delta(X_0) - \phi_{T-t}^\otimes \delta(X_0), \delta(X_0) \rangle + 2\Re \int_0^T dt \langle (\phi_{T-t}^\otimes - id) \delta(X_0), \delta(X_0 - \phi_t(X_0)) \rangle \\ &\leq \int_0^T dt \|\Delta^{1/2}(\phi_t X_0 - X_0)\|_2^2 + T^2 \omega / 4 \|\Delta^{1/2}(X_0)\|_2^2 \\ &+ \int_0^T dt (T - t) \|\Delta^{\otimes 1/2}(\delta(X_0))\|_2^2 \\ &+ \int_0^T dt \sqrt{t(T - t)} (\|\Delta(X_0)\|_2^2 + \omega \|\Delta^{1/2}(X_0)\|_2^2)^{1/2} \|\Delta^{\otimes 1/2}(\delta(X_0))\|_2 \\ &= \int_0^T dt \|\Delta^{1/2}(\phi_t X_0 - X_0)\|_2^2 + T^2 \omega / 4 \|\Delta^{1/2}(X_0)\|_2^2 \\ &+ \frac{T^2}{2} \left(\|\Delta^{\otimes 1/2}(\delta(X_0))\|_2^2 + \frac{\pi}{4} (\|\Delta(X_0)\|_2^2 + \omega \|\Delta^{1/2}(X_0)\|_2^2)^{1/2} \|\Delta^{\otimes 1/2}(\delta(X_0))\|_2 \right). \end{aligned}$$

The inequality comes from $\Gamma_0(\omega)c$ and several uses of the spectral theorem applied in the form $\langle (id - \phi_t)^i x, x \rangle \leq \langle \frac{t}{2} \Delta x, x \rangle$ ($i = 1$ or 2).

Finally it remains to compute the last line using the spectral theorem for Δ :

$$\begin{aligned} \int_0^T dt \|\Delta^{1/2}(\phi_t(X_0) - X_0)\|_2^2 &= 4\|\phi_{T/2}(X_0)\|_2^2 - 3\|X_0\|^2 - \|\phi_T(X_0)\|_2^2 + T\|\Delta^{1/2}(X_0)\|_2^2 \\ &= 2\langle (\phi_T - id + T\Delta/2)(X_0), X_0 \rangle - \|\phi_T(X_0) - X_0\|_2^2 \\ &\leq \frac{T^2}{4} \|\Delta(X_0)\|_2^2 - \|\phi_T(X_0) - X_0\|_2^2, \end{aligned}$$

Putting everything together this concludes to :

$$\begin{aligned} \|X_t^\epsilon - X_0 - (1 - \epsilon)\delta(X_0)\#S_t\|_2^2 &\leq \frac{t^2}{4}\|\Delta(X_0)\|_2^2 + (e^{\omega t} - 1)\|X_0\|_2^2 + (1 - (1 - \epsilon)^2)\|\phi_T(X_0) - X_0\|_2^2 \\ &\quad + \frac{t^2}{2} \left(\|\Delta^{\otimes 1/2}(\delta(X_0))\|_2^2 + \frac{\pi}{4}(\|\Delta(X_0)\|_2^2 + \omega\|\Delta^{1/2}(X_0)\|_2^2)^{1/2}\|\Delta^{\otimes 1/2}(\delta(X_0))\|_2 \right) \\ &\quad + t \sup_{[0,t]} (\|\Delta^{1/2}(\phi_s X_0)\|_2^2 - (1 - \epsilon)^2\|\delta(\phi_s X_0)\|_2^2) + t^2\omega/4\|\Delta^{1/2}(X_0)\|_2^2. \end{aligned}$$

We easily obtain the limit case $\epsilon = 0$ using the limit in $C^0([0, T], (L^2(M), \sigma(L^2(M), L^2(M))))$.

Step 3: Under the assumptions of (ii), with $\mathcal{B}_{2,\delta}^a$ depending on our fixed $T > 0$, and for $X_0 \in D(\Delta)$, there exists a unique mild solution X_t of (2) which is the weak limit in $\mathcal{B}_{2,\delta}^a$ and strong limit in $\mathcal{B}([0, T], L^2(M))$ of the solution X_t^ϵ of step one. Moreover, this solution satisfies the two first inequalities of (ii) in the theorem.

Consider $\epsilon > 0$ like in step 1. In case $\delta \neq \tilde{\delta}$, we don't know $X_t^\epsilon \in D(\Delta)$ since we have only a mild solution, we have to circumvent this trouble for computational purposes.

Applying the first part of step 1 with δ replaced by $\eta_\beta^\otimes \delta$, we get a solution $X_t^{\epsilon,\beta}$ in $\mathcal{B}_{2,\Delta^{1/2}}^a$ and since by proposition 5 (5) and the argument in step one, $\gamma(X_t^{\epsilon,\beta}) \in \mathcal{B}_{2,\Delta^{1/2}}^a$ we deduce $\eta_\beta \gamma(X_t^{\epsilon,\beta}) \in \mathcal{B}_{2,\Delta^{3/2}}^a$. As a consequence if $X_0 \in D(\Delta)$ we get as in step 1, $X_t^{\epsilon,\beta} \in \mathcal{B}_{2,\Delta^{3/2}}^a$.

We can now compute for our solution $X_t^{\epsilon,\beta}$. We can apply Proposition 5 (6) in case $B = \tilde{\delta}$, $\alpha = \alpha' = \beta$ and the variant of (4) valid for $x = X_0 \in D(\Delta^{1/2})$: $\|\tilde{\delta}\phi_t(x)\|_2^2 = \|\tilde{\delta}x\|_2^2 - \int_0^t \Re \langle \tilde{\delta}\Delta\phi_s(x), \tilde{\delta}\phi_s(x) \rangle ds$. Using also orthogonality from lemma 3 but for $\tilde{\delta}$, we get :

$$\|\tilde{\delta}(X_t^{\epsilon,\beta})\|_2^2 = \|\tilde{\delta}(X_0^\epsilon)\|_2^2 + (1 - \epsilon)^2 \|\tilde{\delta}\eta_\beta \int_0^t \delta(X_s^{\epsilon,\beta})\#dS_s\|_2^2 - \int_0^t \Re \langle \tilde{\delta}\Delta X_s^{\epsilon,\beta}, \tilde{\delta}X_s^{\epsilon,\beta} \rangle ds.$$

We have thus shown :

$$\begin{aligned} \|\tilde{\delta}(X_t^{\epsilon,\beta})\|_2^2 &= \|\tilde{\delta}(X_0^\epsilon)\|_2^2 \\ &\quad + (1 - \epsilon)^2 \|(\eta_\beta^\otimes \tilde{\delta} + \tilde{\mathcal{H}}_\beta) \int_0^t \delta(X_s^{\epsilon,\beta})\#dS_s\|_2^2 - \int_0^t \Re \langle \Delta^\otimes \circ \tilde{\delta}(X_s^{\epsilon,\beta}) - \mathcal{H}(\tilde{\delta}(X_s^{\epsilon,\beta})), \tilde{\delta}X_s^{\epsilon,\beta} \rangle ds. \end{aligned}$$

We used the identities of assumption f) and of lemma 2 about $\tilde{\delta}\Delta$ and $\tilde{\delta}\eta_\beta$ justified since almost surely in s $X_s^{\epsilon,\beta} \in D(\Delta^{3/2})$ and because via lemma 3 (i) we know $\int_0^t \delta(X_s^{\epsilon,\beta})\#dS_s \in$

$D(\tilde{\delta})$. We deduce :

$$\begin{aligned}
& \|\tilde{\delta}(X_t^{\epsilon,\beta})\|_2^2 \\
& \leq \|\tilde{\delta}(X_0^\epsilon)\|_2^2 + (1-\epsilon)^2 \int_0^t \|\tilde{\delta}^\otimes \delta(X_s^{\epsilon,\beta})\|_2^2 ds + 2\Re\langle \tilde{\delta}\eta_\beta \int_0^t \delta(X_s^{\epsilon,\beta})\#dS_s, \tilde{\mathcal{H}}_\beta \int_0^t \delta(X_s^{\epsilon,\beta})\#dS_s \rangle \\
& \quad - \int_0^t \Re\langle \Delta^\otimes \circ \tilde{\delta}(X_s^{\epsilon,\beta}) - \mathcal{H}(\tilde{\delta}(X_s^{\epsilon,\beta})), \tilde{\delta}X_s^{\epsilon,\beta} \rangle ds \\
& \leq \|\tilde{\delta}(X_0^\epsilon)\|_2^2 + \int_0^t 2C\|\tilde{\delta}(X_s^{\epsilon,\beta})\|_2^2 ds + 2\Re\langle \tilde{\delta}\eta_\beta \int_0^t \delta(X_s^{\epsilon,\beta})\#dS_s, \tilde{\mathcal{H}}_\beta \int_0^t \delta(X_s^{\epsilon,\beta})\#dS_s \rangle \\
& \leq \|\tilde{\delta}(X_0^\epsilon)\|_2^2 + \int_0^t (2C + 2\frac{C^4}{\beta}(\omega + 2\beta))\|\tilde{\delta}(X_s^{\epsilon,\beta})\|_2^2 ds.
\end{aligned}$$

In the first line we used η_β contractive after computing the first scalar product. In the second line we used assumption h to cancel one term and the bound $\|\mathcal{H}\| \leq C$. In the last line we used assumption g, the bound on $\tilde{\mathcal{H}}_\beta$ from lemma 2 and $\|\tilde{\delta}\eta_\beta\| \leq C\sqrt{\omega + 2\beta}$ already used there.

Applying Gronwall's lemma, we got (for $\beta \geq 1$):

$$\|\tilde{\delta}(X_t^{\epsilon,\beta})\|_2^2 \leq \|\tilde{\delta}(X_0^{\epsilon,\beta})\|_2^2 e^{(6+2\omega)C^4 t}.$$

As a consequence, we get a weak limit point $X_t^{\epsilon,\infty}$ in $\mathcal{B}_{2,\delta}^a$. Let us show such a limit point is a solution of (2ϵ) in $\mathcal{B}_{2,\delta}^a$, giving by uniqueness $X_t^{\epsilon,\infty} = X_t^\epsilon$, and the fact that the weak limit point is a limit. Of course it suffices to show the equation weakly, the only non trivial limit is the stochastic integral, but since $\delta X_t^{\epsilon,\beta}$ is bounded it is easy to remove η_β on the other side of the scalar product, and then to use weak convergence of $X_t^{\epsilon,\beta}$ in $\mathcal{B}_{2,\delta}^a$. We also get a corresponding inequality a.e. for the limit by seeing the inequality weakly in $L^2([0, T])$.

As is usual, if we are able to prove bounds in $D(\delta)$, we can also deduce $\|\cdot\|_2$ Cauchy property. Using (5) after using the SDE and the common initial conditions, we also get (for $0 < \epsilon, \eta < 1$) :

$$\begin{aligned}
& \|X_t^\epsilon - X_t^\eta\|_2^2 = \|\gamma((1-\epsilon)X_t^\epsilon - (1-\eta)X_t^\eta)\|_2^2 \\
& = - \int_0^t \|\Delta^{1/2}\gamma((1-\epsilon)X_s^\epsilon - (1-\eta)X_s^\eta)\|_2^2 ds + \int_0^t \|\delta((1-\epsilon)X_s^\epsilon - (1-\eta)X_s^\eta)\|_2^2 ds \\
& \leq - \int_0^t \|\Delta^{1/2}(X_s^\epsilon - X_s^\eta)\|_2^2 ds + \int_0^t \|\delta(X_s^\epsilon - X_s^\eta)\|_2^2 + 12\max(\epsilon, \eta)\max(\|\delta(X_s^\epsilon)\|_2, \|\delta(X_s^\eta)\|_2)^2 ds.
\end{aligned}$$

In the last line we used an elementary bound on the second integral expanding the scalar products with $(1-\epsilon)X_s^\epsilon - (1-\eta)X_s^\eta = (X_s^\epsilon - X_s^\eta) + (\eta X_s^\eta - \epsilon X_s^\epsilon)$ and again the SDE with same initial condition on the first integral. Using assumption c) and our bound on $\|\delta(X_s^\epsilon)\|_2$, one gets :

$$\begin{aligned}
& \|X_t^\epsilon - X_t^\eta\|_2^2 \leq \int_0^t \omega\|(X_s^\epsilon - X_s^\eta)\|_2^2 + 12\max(\epsilon, \eta)\|\tilde{\delta}(X_0^{\epsilon,\beta})\|_2^2 e^{(6+2\omega)C^4 s} ds \\
& \leq 12\max(\epsilon, \eta)\|\tilde{\delta}(X_0^{\epsilon,\beta})\|_2^2 \frac{e^{(6+2\omega)C^4 t + \omega t}}{(6+2\omega)C^4}.
\end{aligned}$$

As noted at the beginning of step 2 we know any (mild) solution of the case $\epsilon = 0$, if it exists satisfies $\|X_t\|_2^2 \leq e^{\omega t} \|X_0\|_2^2$, giving especially uniqueness.

We have thus obtained strong convergence on X_t^ϵ in $\mathcal{B}([0, T], L^2(M))$ by Cauchy property. We have also boundedness of X_t^ϵ in $B_{2,\delta}^a$, which gives by weak compactity a limit up to extraction when $\epsilon \rightarrow 0$. Once we will have proved that any such limit point is a mild solution with $\epsilon = 0$, uniqueness (of the solution thus of the limit point) will get that in fact X_t^ϵ weakly converges in $\mathcal{B}_{2,\delta}^a$ to the newly found solution X_t . Since we have already noticed weak continuity of Stochastic convolution, we are in fact done (for proving that any limit point is a mild solution).

Finally we conclude the proof of the part (ii) of the Theorem, by considering $\eta_\alpha(X_0)$ as initial condition of a solution $X_{t,\alpha}$, in case we have only $X_0 \in D(\Delta^{1/2})$ (and not anymore $D(\Delta)$) and letting go $\alpha \rightarrow \infty$ with the same weak limit arguments we show that $X_{t,\alpha}$ converges weakly in $\mathcal{B}_{2,\Delta^{1/2}}^a$ to X_t . Moreover, note for further use that we have also strong convergence of $X_{t,\alpha}$ to X_t in $\mathcal{B}([0, T], L^2(M))$ by the following inequality (proved as above for the Cauchy property, except we don't have the same initial conditions anymore, but more cancellations) :

$$\begin{aligned} \|X_{t,\alpha} - X_{t,\beta}\|_2^2 &= \|\phi_t(X_{0,\alpha} - X_{0,\beta})\|_2^2 + \|\gamma(X_{t,\alpha} - X_{t,\beta})\|_2^2 \\ &= \|\phi_t(X_{0,\alpha} - X_{0,\beta})\|_2^2 - \int_0^t \|\Delta^{1/2} \gamma(X_{s,\alpha} - X_{s,\beta})\|_2^2 ds + \int_0^t \|\delta(X_{s,\alpha} - X_{s,\beta})\|_2^2 ds \\ &\leq \|X_{0,\alpha} - X_{0,\beta}\|_2^2 - \int_0^t \|\Delta^{1/2}(X_{s,\alpha} - X_{s,\beta})\|_2^2 - \|\Delta^{1/2} \phi_s(X_{0,\alpha} - X_{0,\beta})\|_2^2 ds \\ &\quad + \int_0^t \|\Delta^{1/2}(X_{s,\alpha} - X_{s,\beta})\|_2^2 + \omega \|X_{s,\alpha} - X_{s,\beta}\|_2^2 ds \\ &\leq e^{\omega t} (\|\eta_\alpha(X_0) - \eta_\beta(X_0)\|_2^2 + T \|\eta_\alpha(\Delta^{1/2} X_0) - \eta_\beta(\Delta^{1/2} X_0)\|_2^2). \end{aligned}$$

□

2. OUR MAIN EXAMPLE : DERIVATION-GENERATOR OF A DIRICHLET FORM

As explained in the introduction, our main case of interest will be when δ is a derivation and $\Delta = \delta^* \delta$ the corresponding generator of a Dirichlet form. Note that in that case it is well known (cf e.g. [9]) ϕ_t and η_α are completely positive contractions on M .

2.1. Preliminaries and notation around zero extensions of a derivation on free Brownian motions.

2.1.1. Setting and extension. Recall $M = W^*(M_0; S_s^{(j)}, 0 \leq s \leq \infty, 0 \leq j \leq N)$ (we will consider only here the case of finitely many derivations and thus free Brownian motions) and $M_t = W^*(M_0; S_s^{(j)}, 0 \leq s \leq t, 0 \leq j \leq N)$.

Let us assume we are given $\partial : D(\partial) \rightarrow HS(M_0)^N \simeq (L^2(M_0) \otimes L^2(M_0))^N \simeq_{1 \otimes O} (L^2(M_0) \otimes L^2(M_0^{op}))^N$ a derivation valued in a direct sum of Hilbert-Schmidt operators over $L^2(M_0)$. As usual the identification of $L^2(M_0) \otimes L^2(M_0)$ to Hilbert-Schmidt operators sends $a \otimes b$ to the finite rank operator $x \mapsto a\tau(bx)$. As real bimodules they are considered with bimodule structure induced by $a(b \otimes c)d = ab \otimes cd$, and real structure $\mathcal{J}(a \otimes b) = b^* \otimes a^*$ corresponding to adjointness of Hilbert-Schmidt operators. We will emphasize the isomorphism $1 \otimes O$ with

$L^2(M_0) \otimes L^2(M_0^{op})$ (coming from traciality) with corresponding bimodule structure when necessary (it is induced by the identity map for $a, b \in M$ $(1 \otimes O)a \otimes b = a \otimes b$ with b seen in M^{op}).

First we consider a notion of equivalence of derivations (corresponding to a strong way of requiring they have the same domain with equivalent norms).

We write $Z_j = (0, \dots, 0, 1 \otimes 1, 0, \dots, 0)$ in $HS(M_0)^N$ the non-zero term lying on the j^{th} component. We also write ∂_j for the j^{th} component in $HS(M_0)^N$ (and we will use freely later this kind of notations). For $U \in L^2(M) \otimes L^2(M^{op})$, $K \in M \overline{\otimes} M^{op}$, we write consistently with our previous notation $U \# K$ the map induced by multiplication in $M \overline{\otimes} M^{op}$. If $U \in L^2(M) \otimes L^2(M)$, we write in this way the map induced by the previous isomorphism : $1 \otimes O(U \# K) := 1 \otimes O(U) \# K$.

Definition 11. Two real derivations $\partial_{(1)}, \partial_{(2)}$ as above defined on the same domain \mathcal{D} are said to be **equivalent** if for every j there exists $i_j, k_j \in [1, N]$ and $I_j, K_j \in M \overline{\otimes} M^{op}$ (without loss of generality $\mathcal{J}(I_j) = I_j, \mathcal{J}(K_j) = K_j$), such that $\forall x \in \mathcal{D}$ $(1 \otimes O)\partial_{(1)j}(x) = (1 \otimes O)\partial_{(2)i_j}(x) \# I_j$, $(1 \otimes O)\partial_{(2)j}(x) = (1 \otimes O)\partial_{(1)k_j}(x) \# K_j$. We say $\partial_{(1)}$ is **reducible** to $\partial_{(2)}$ ϖ_1 -regularly with respect to $\partial_{(3)}$ (another real derivation closable $L^2(M_0) \rightarrow L^{\varpi_1}(M_0 \otimes M_0)$, $\varpi_1 \geq 2$), with coefficients (k_j, i_j, K_j, I_j) , if they are equivalent as above with I_j invertible in $M \overline{\otimes} M^{op}$, and $I_j^{-1} \in D(\overline{1 \otimes \partial_{(3)}^{\varpi_1}}) \cap D(\overline{\partial_{(3)} \otimes 1}^{\varpi_1})$.

Note that if $\partial_{(1)}, \partial_{(2)}$ are closable and equivalent so are their closures.

Domains of closures will be considered in this L^2 setting, $D(\partial) \subset M_0$ is a weakly dense $*$ -subalgebra. We will really soon impose conditions making ∂ closable as an unbounded operator from $L^2(M_0, \tau) \rightarrow HS(M_0)^N$, and real (i.e. we have the relation $\partial(x)^* = \partial(x^*)$ with the adjoint of Hilbert-Schmidt operators in each component and as a consequence $\langle \partial(x), y \partial(z) \rangle = \langle \partial(z^*) y^*, \partial(x^*) \rangle, \forall x, y, z \in D(\partial)$). After extending it to a closed derivation $\bar{\partial}$ on M we will be interested in the corresponding generator of a Dirichlet form $\Delta = \delta^* \bar{\delta}$. This part will find realistic assumptions on ∂ to get $\Gamma_1(\omega, C)$ and thus to be able to apply our general theory.

Suppose also that $\mathcal{J}_j := \partial^*(Z_j) \in L^2(M_0)$ is well defined for all $j \in [1, N]$. We have a well-known lemma (identical to Proposition 4.1 in [53] which is valid for any real derivation of the kind considered above, as pointed out after Proposition 6.2 in [54]) :

Lemma 12. *With the conditions above (except the condition closable which can be deduced from the other conditions via the lemma, i.e. ∂ real densely defined derivation with $\mathcal{J}_j := \partial^*(Z_j) \in L^2(M_0)$), $(D(\partial) \otimes_{alg} D(\partial))^N$ is contained in $D(\partial^*)$ and we have :*

$$\partial_j^*(a \otimes b) := \partial^*(a Z_j b) = a \mathcal{J}_j b - (1 \otimes \tau)[\partial_j(a)]b - a(\tau \otimes 1)[\partial_j(b)].$$

Moreover (see e.g. [11] Remark 7, using mainly [14], cf. also [32]), $\bar{\partial}|_{M_0 \cap D(\bar{\partial})}$ defines a derivation (noted ∂^∞ on the $*$ -algebra $M_0 \cap D(\bar{\partial})$), closed as an unbounded operator $M_0 \rightarrow HS(M_0)^N$. Finally (see e.g. proposition 6 in [11]), for any $Z \in D(\bar{\partial}) \cap M_0$, there exists a sequence $Z_n \in D(\partial)$ with $\|Z_n\| \leq \|Z\|$, $\|Z_n - Z\|_2, \|\partial(Z_n) - \bar{\partial}(Z)\|_2 \rightarrow 0$; the statements not involving ∂^* also hold for a derivation equivalent to ∂ . ■

Consider also $D(\delta) = D(\partial) * \mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle \subset M$, the algebra generated by $S_s^{(j)}$ and $D(\partial)$ (thus $D(\delta)$ is a weakly dense $*$ -subalgebra of M). Define $\delta : D(\delta) \rightarrow HS(L^2(M))^N$ the unique derivation such that $\delta(x) = \partial(x)$ if $x \in D(\partial)$ and $\delta(S_t^{(j)}) = 0$ for all t . Then,

clearly $\mathcal{J}_j = \delta^*(Z_j) \in L^2(M_0) \subset L^2(M)$ (see e.g. [44, Example 2.4]), and using the lemma above, δ is also closable (since δ^* is densely defined). δ is thus a closable real derivation, like ∂ . If we start instead from $\partial_{(1)}$ equivalent to ∂ we get in the same way a closable $\delta_{(1)}$ equivalent to δ . We may sometimes write $\delta^\infty : M \cap D(\bar{\delta}) \rightarrow HS(L^2(M))^N$ the analog derivation defined in the previous lemma (when we want to emphasize the domain). We will write $\Delta = \delta^* \bar{\delta}$ the associated generator of a completely Dirichlet form, ϕ_t the semigroup generated by $-1/2\Delta$, $\eta_\alpha = \frac{\alpha}{\alpha + \Delta}$ the “resolvent map” associated, as before. As we already pointed out, they induce completely positive contraction on M .

We thus only assumed assumption 0 or even 0':

Assumption 0 : (a) $\partial : D(\partial) \rightarrow HS(M_0)^N$ real derivation $D(\partial) \subset M_0$ weakly dense *-subalgebra

(b) $\mathcal{J}_j := \partial^*(Z_j) \in L^2(M_0)$ is well defined for all $j \in [1, N]$, and δ is an extension by 0 on free Brownian motions : $\delta(x) = \partial(x)$ if $x \in D(\partial)$ and $\delta(S_t^{(j)}) = 0$ for all t .

Assumption 0' (resp $0'^{\partial_{(3)}, \varpi_1}$) : (a) $\partial : D(\partial) \rightarrow HS(M_0)^N$ real derivation $D(\partial) \subset M_0$ weakly dense *-subalgebra

(b) ∂ is equivalent (resp. reducible ϖ_1 -regularly with respect to $\partial_{(3)}$) to $\tilde{\partial}$ satisfying assumption 0 and δ is an extension by 0 on free Brownian motions : $\delta(x) = \partial(x)$ if $x \in D(\partial)$ and $\delta(S_t^{(j)}) = 0$ for all t .

This subsection will mainly develop general consequences of this assumption 0, giving at the end Γ_0 . Assumptions with primes are technical and will be used mainly in section 2.1.5 and its preliminaries and sequels, assumption with ϖ_1 -regularity will be used in the key proposition 24 through lemma 25. Note that any prime variant is implied by a non-prime variant with the same derivation equivalent via $1 \otimes 1$. We may sometimes use stronger assumptions ($p, q, \varpi_1 \geq 2$):

Assumption 0_p (resp $0'_p, 0_p'^{\partial_{(3)}, \varpi_1}$) : (a) $\partial : D(\partial) \rightarrow (L^p(M_0) \otimes_{alg} L^p(M_0))^N \rightarrow HS(M_0)^N$ real derivation $D(\partial) \subset M_0$ weakly dense *-subalgebra
(b) δ satisfies 0 (resp $0', 0'^{\partial_{(3)}, \varpi_1}$)

Assumption 0_p^a (resp $0_p^{a'}, 0_p^{a'\partial_{(3)}, \varpi_1}$) : (a) $\partial : D(\partial) \rightarrow (L^p(M_0 \otimes M_0))^N \rightarrow HS(M_0)^N$ (resp. $\partial : D(\partial) \rightarrow (L^p(M_0) \hat{\otimes} L^p(M_0))^N \rightarrow HS(M_0)^N$) real derivation $D(\partial) \subset M_0$ weakly dense *-subalgebra

(b) δ satisfies 0 (resp $0', 0_p'^{\partial_{(3)}, \varpi_1}$)

Note that real corresponds to the corresponding adjoint map on the algebraic tensor product or in L^2 setting and we always consider closures or equivalence for the derivation with value in the L^2 setting induced by the canonical map written above.

Assumption $0_{p,q}$ (resp $0'_{p,q}, 0_{p,q}'^{\partial_{(3)}, \varpi_1}, 0_{p,q}^a, 0_{p,q}^{a'}, 0_{p,q}^{a'\partial_{(3)}, \varpi_1}$) : Assumption 0_p (resp $0'_p, 0_p'^{\partial_{(3)}, \varpi_1}, 0_p^a, 0_p^{a'}, 0_p^{a'\partial_{(3)}, \varpi_1}$) and (c) $\mathcal{J}_j := \partial^*(Z_j) \in L^q(M_0)$ (resp. for the corresponding equivalent derivation given by $0', 0'^{\partial_{(3)}, \varpi_1}$ in cases with primes)

2.1.2. *Useful L^1 -closures.* Here we assume assumption 0'.

We will also define following [35] 1.4, an analog of Δ , $\Delta^1 : M \rightarrow L^1(M, \tau)$ (there noted Ψ), by

$$(11) \quad D(\Delta^1) = \{x \in D(\bar{\delta}) \cap M \mid y \mapsto \langle \bar{\delta}(x), \bar{\delta}(y) \rangle \text{ extends to a normal linear functional on } M\}$$

$\Delta^1(x)$ is defined as the adjoint of the Radon-Nikodym derivative of the preceding linear functional $y \mapsto \langle \bar{\delta}(x), \bar{\delta}(y) \rangle$, i.e. $\langle \Delta^1(x), y \rangle := \tau(\Delta^1(x)^* y) = \langle \bar{\delta}(x), \bar{\delta}(y) \rangle$ (note the antilinear duality bracket consistent with scalar products).

Likewise we can define $\delta^{*1} : (L^2(M) \otimes L^2(M))^N \rightarrow L^1(M, \tau)$, by

$$D(\delta^{*1}) = \{U \in (L^2(M) \otimes L^2(M))^N \mid y \mapsto \langle U, \bar{\delta}(y) \rangle\}$$

extends to a normal linear functional on M).

$\delta^{*1}(U)$ is defined as the adjoint of the Radon-Nikodym derivative of the preceding linear functional $y \mapsto \langle U, \bar{\delta}(y) \rangle$. By the very definition, we see that for any $x \in D(\Delta^1)$, $\bar{\delta}(x) \in D(\delta^{*1})$ and $\Delta^1(x) = \delta^{*1}\bar{\delta}(x)$. Moreover, we see obviously that δ^{*1} is a closed densely defined operator (using $D(\delta^*) \subset D(\delta^{*1})$ and $\bar{\delta}|_{M \otimes D(\bar{\delta})}$ is a densely defined formal adjoint.). Note the following elementary lemma (using mainly the fact that δ^∞ is a derivation) :

Lemma 13. *$D(\Delta^1)$ is a $*$ -subalgebra of M containing $D(\Delta) \cap M$, and for any $x, y \in D(\Delta^1)$:*

$$\Delta^1(xy) = \Delta^1(x)y + x\Delta^1(y) - 2 \sum_{i=1}^N m \circ (1 \otimes (\tau \circ m) \otimes 1)(\bar{\delta}_i(x) \otimes \bar{\delta}_i(y)),$$

where m denote the multiplication map $L^2(M) \hat{\otimes} L^2(M) \rightarrow L^1(M)$. Finally, for any $x, y \in D(\Delta^1)$: $\langle \Delta^1(x), y \rangle = \langle x, \Delta^1(y) \rangle$.

Proof. Take $x, y \in D(\Delta^1)$, $z \in M \cap D(\delta)$,
thus

$$\begin{aligned} \langle \delta(xy), \delta(z) \rangle &= \langle \delta(x)y, \delta(z) \rangle + \langle x\delta(y), \delta(z) \rangle \\ &= \langle \delta(x), \delta(z)y^* \rangle + \langle \delta(y), x^*\delta(z) \rangle \\ &= \langle \delta(x), \delta(zy^*) \rangle + \langle \delta(y), \delta(x^*z) \rangle - \langle \delta(x), z\delta(y^*) \rangle - \langle \delta(y), \delta(x^*)z \rangle \\ &= \langle \Delta^1(x)y, z \rangle + \langle x\Delta^1(y), z \rangle - \langle \delta(y)z^*, \delta(x^*) \rangle - \langle \delta(y), \delta(x^*)z \rangle \\ &= \langle \Delta^1(x)y, z \rangle + \langle x\Delta^1(y), z \rangle - 2 \sum_i \text{Tr}(\delta_i(y)^* \circ \delta_i(x)^* z) \\ &= \langle \Delta^1(x)y, z \rangle + \langle x\Delta^1(y), z \rangle - 2\tau(zm(\sum_i \delta_i(x) \circ \delta_i(y))^*). \end{aligned}$$

In the fourth line, we used the definition of Δ^1 and the fact δ is a real derivation. We used at the next to last line the identification of $L^2 \otimes L^2$ with Hilbert Schmidt operators and the Trace on trace class, and the relation $\delta_i(x)^* = \delta_i(x^*)$ with the adjoint of Hilbert-Schmidt operators coming from the fact we have a real derivation. At the last line we used the multiplication map to $L^1(M)$, induced by $m(a \otimes b) = ab$.

This proves the domain property and the equation. \square

We will also need an extension $\overline{\Delta^1} : L^2(M) \rightarrow L^1(M)$. But the last equality of the previous lemma especially shows that $\Delta|_{D(\Delta) \cap M} : M \rightarrow L^2(M)$ is a $(\sigma$ -weakly) densely defined formal adjoint of $\Delta^1 : L^2(M) \rightarrow L^1(M)$, thus this operator is closable. And moreover, for any $x \in D(\Delta) \cap M$, $y \in D(\overline{\Delta^1})$, $\langle \Delta(x), y \rangle = \langle x, \overline{\Delta^1}(y) \rangle$.

Note the following elementary lemma, using $M \cap D(\Delta)$ is a core for Δ (thanks to stability of M by ϕ_t) :

Lemma 14. *For any $x, y \in D(\Delta)$ with either x or y in M , then $xy \in D(\overline{\Delta^1})$:*

$$\overline{\Delta^1}(xy) = \Delta(x)y + x\Delta(y) - 2 \sum_{i=1}^N m \circ (1 \otimes \tau \circ m \otimes 1)(\bar{\delta}_i(x) \otimes \bar{\delta}_i(y)),$$

where m denotes the multiplication map $L^2(M) \hat{\otimes} L^2(M) \rightarrow L^1(M)$. ■

We end this section with an extension of $\delta : L^2 \rightarrow L^{1+\epsilon}$. We also write $1 \otimes O$ for the isometry $L^p(M) \hat{\otimes} L^p(M) \rightarrow L^p(M) \hat{\otimes} L^p(M^{op})$.

Lemma 15. *Assume $0_{p,q}^a$, $q, p \geq 2$, then, and $r \leq \infty$ such that $1/r \leq \min(1 - 1/q, 1 - 1/p)$, $s \in (1, p]$, $\delta : L^r(M) \rightarrow (L^s(M \otimes M))^N$ (with domain $D(\delta)$) is closable, we write $\delta^{r,s}$ this closure, δ^s if $r = 2$. Moreover for any $y \in D(\delta^{r',s'})$ with either $x \in D(\delta^{r,s})$, or $x \in D(\bar{\delta}) \cap M$ (then take $r = \infty$, $s = 2$ and write below $\delta^{r,s} = \bar{\delta}$ abusively) or $x \in D(\delta)$ (then take $r = \infty$, $s = p$ and " $\delta^{r,s} = \delta$ " abusively), such that there exists r'', s'' satisfying the constraints for r, s above and $1/r'' \geq 1/r + 1/r'$ and $1/s'' \geq \max(1/r' + 1/s, 1/r + 1/s')$ then $xy \in D(\delta^{r'',s''})$ with $\delta^{r'',s''}(xy) = \delta^{r,s}(x)y + x\delta^{r',s'}(y)$. Assume only $0_{p,q}^{a'}$, then the same is true for $1 \otimes O\delta$ and we write $\delta^{O;r,s} : L^r(M) \rightarrow (L^s(M \otimes M^{op}))^N$ the closure.*

Proof. Since $M_0 \otimes_{alg} M_0^{op}$ is dense in the dual of L^s , the densely defined adjoint is already given by lemma 12 (for $r = \infty$, the pre-adjoint between preduals). Note, the assumption on r, q, p gives the right space for the adjoint in $L^q + L^p \subset L^{r/(r-1)}$. By Hölder inequality, the derivation property goes to the limit. In case of assumption $0_{p,q}^{a'}$, $1 \otimes O\delta : D(\delta) \rightarrow (L^p(M \otimes M^{op}))^N \rightarrow (L^2(M \otimes M^{op}))^N$ and equivalence gives that the equivalent derivation is valued in $(L^p(M \otimes M^{op}))^N$ so that the reasoning before can be applied to it, and equivalence gives it back for δ . □

For $U \in L^r \otimes L^{r'}$, $Z \in L^q$, we note $U \# Z \in L^p$ with $1/p = 1/r + 1/r' + 1/q$ the usual extension of $a \otimes b \# Z = aZb$ given by Hölder inequality (for appropriate ranges of r, r', q), $mU = U \# 1$ the corresponding multiplication map ($q = \infty$ here). With those notations, the following limit variant of lemma 12 is obvious.

Lemma 16. *Assume δ satisfy $0_{p,q}^a$ with $q, p \geq 2$, $r \in [2, \infty]$, $s \in (1, p]$ $1/s' := \max(1/s + 1/r, 2/r + 1/q) \leq 1$. Consider $U \in D(\delta^{r,s}) \otimes_{alg} D(\delta^{r,s})$.*

*Then $U \in D(\delta^{*1})$ and $\delta_j^{*1}U = U \# \mathcal{J}_j - m(1 \otimes \tau \otimes 1)(\delta_j^{r,s} \otimes 1 + 1 \otimes \delta_j^{r,s})(U) \in L^{s'}$.* ■

2.1.3. Lemmas about the extension. Here again we only assume $0'$.

We can consider $(\delta \otimes 1) \oplus (1 \otimes \delta) : L^2(M) \otimes L^2(M) \rightarrow (L^2(M) \otimes L^2(M) \otimes L^2(M))^{2N}$, (later abbreviated $\delta \otimes 1 \oplus 1 \otimes \delta$ or δ^{\otimes}) which is easily seen to be densely defined on $D(\delta) \otimes_{alg} D(\delta)$, and closable (with an explicit densely defined adjoint coming from lemma 12 in case of assumption 0). We will write $\Delta^{\otimes} := (\delta \otimes 1 \oplus 1 \otimes \delta)^*(\delta \otimes 1 \oplus 1 \otimes \delta) = \overline{\Delta \otimes 1 + 1 \otimes \Delta}$, which is thus a densely defined closed self-adjoint positive operator. It can be seen, as stated above, to be equal to the closure of $\Delta \otimes 1 + 1 \otimes \Delta$ (defined on $D(\Delta) \otimes_{alg} D(\Delta)$, using the stability of this space by $\phi_t \otimes \phi_t$, or rather more the regularization effect, implying this is a core of the previous closed operator).

Recall $\#(S_t^i - S_s^i)/\sqrt{t-s} : L^2(M_s) \otimes L^2(M_s) \rightarrow L^2(M)$ is the standard isometry extending $(a \otimes b) \#(S_t^i - S_s^i) = a(S_t^i - S_s^i)b$. Likewise, we define $\#_j$ extending $(a \otimes b \otimes c) \#_1(S_t^i - S_s^i) = a(S_t^i - S_s^i)b \otimes c$ and $(a \otimes b \otimes c) \#_2(S_t^i - S_s^i) = a \otimes b(S_t^i - S_s^i)c$, $a, b, c \in M_s$.

Corollary 17. *For any $U \in D(\Delta^\otimes) \cap L^2(M_s) \otimes L^2(M_s)$, then $U\#(S_t^i - S_s^i) \in D(\Delta)$ ($t \geq s$) and :*

$$\Delta(U\#(S_t^i - S_s^i)) = \Delta^\otimes(U)\#(S_t^i - S_s^i).$$

Moreover, for any $U \in L^2(M_s) \otimes L^2(M_s)$, $U\#(S_t^i - S_s^i) \in D(\delta)$ if and only if $U \in D(\overline{\delta \otimes 1 \oplus 1 \otimes \delta})$, and we also have

$$\delta(U\#(S_t^i - S_s^i)) = \overline{\delta \otimes 1}(U)\#_2(S_t^i - S_s^i) + \overline{1 \otimes \delta}(U)\#_1(S_t^i - S_s^i).$$

As a consequence, such an element is orthogonal to any $L^2(M_0 \otimes M_0)$ (as claimed in assumption d)).

Proof. Consider $U \in (D(\Delta) \cap M_s) \otimes_{alg} (D(\Delta) \cap M_s)$, and by linearity even $U = a \otimes b$, then by lemma 13, we have $U\#(S_t^i - S_s^i) \in D(\Delta)$ and the formula comes from the formula there (applied twice and using freeness to cancel the other terms). The density remark before the proof and the isometry $\cdot\#(S_t^i - S_s^i)/\sqrt{t-s} : L^2(M_s) \otimes L^2(M_s) \rightarrow L^2(M)$ conclude the general case.

The second property comes from δ a derivation starting with the case $U \in D(\delta) \otimes_{alg} D(\delta) \cap M_s \otimes M_s$, and using δ closed for the if part and in order to extend the formula. The only if part uses δ is defined first on $D(\partial) * \mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle$. and the fact we can take the approximation of $U\#(S_t^i - S_s^i)$ in the image of $M_s \otimes_{alg} M_s$ by $\cdot\#(S_t^i - S_s^i)$ (using freeness and the derivation property on the free product above to get the projection of a first approximation on the set above is dominated for the norm of δ by the first one). \square

We will need a result to find a core for $\bar{\delta}$ from one for $\bar{\partial}$, this is given by the following :

Lemma 18. *Let $B \subset M$ be a $*$ -algebra, core for $\bar{\partial} : L^2(M_0) \rightarrow L^2(M_0 \otimes M_0)$, then $B * \mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle$ is a core of $\bar{\delta} : L^2(M) \rightarrow L^2(M \otimes M)$. Analogously if $B \subset M$ be a $*$ -algebra, core for $\partial^* \bar{\partial}$, then $B * \mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle$ is a core for $\delta^* \delta$.*

Proof. We have to show that for any $x \in D(\bar{\delta})$, $\epsilon > 0$ we have a $x_\epsilon \in B * \mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle$ such that $\|x - x_\epsilon\|_2, \|\bar{\delta}(x - x_\epsilon)\|_2 \leq \epsilon$. First, by definition $D(\bar{\delta}) = D(\partial) * \mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle$ is a core for $\bar{\delta}$. Thus take first a $y_\epsilon \in D(\partial) * \mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle$ with $\|x - y_\epsilon\|_2, \|\bar{\delta}(x - y_\epsilon)\|_2 \leq \epsilon/2$ and write y_ϵ as a finite sum of M products $x_1 x_2 \dots x_p$ of elements of $D(\partial)$ or $\mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle$. Then for each such product of p factors take a bound $R \geq 1$ on the $\|\cdot\|_\infty$ -norm and $\|\partial(\cdot)\|_2$ of each factor. Inasmuch as we have assumed B is a core for $\bar{\partial}$, use the last remark of lemma 12 (where B replaces $D(\partial)$, using the equivalence of σ -strong convergence and $\|\cdot\|_2$ -convergence on bounded subsets of M and the σ -strong continuity of the actions of M on $HS(M)$) to approximate each factor x_i coming from $D(\partial)$ by a term $x_{i,\epsilon}$ of B , bounded in $\|\cdot\|_\infty$ by R , close of x_i under $\|\partial(\cdot)\|_2$ and $\|\cdot\|_2$ say as close as $\epsilon/2Mp^2R^{p-1}$ and such that we also have $\|(x_i - x_{i,\epsilon})x_{i+1} \dots x_{i+k}\bar{\partial}(x_{i+k+1})\|_2, \|\bar{\partial}(x_{i-k-1})x_{i-k} \dots x_{i-1}(x_i - x_{i,\epsilon})\|_2 \leq \epsilon/2Mp^2R^{p-1}$, for all suitable k , using the strong convergence for the above finite number of vectors. Moreover if $x_i \in \mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle$, take $x_{i,\epsilon} = x_i$. To obtain x_ϵ , replace in y_ϵ the M products $x_1 \dots x_p$ by $x_{1,\epsilon} \dots x_{p,\epsilon}$ as above. This x_ϵ fit our requirements. To detail a bit, for instance, the hardest approximation to prove, the one for $\delta(x_\epsilon - y_\epsilon)$, requires to consider M terms $\delta(x_1 \dots x_p - x_{1,\epsilon} \dots x_{p,\epsilon})$, and thus p products after application of Leibniz rule for such a

term, typically (in the worst case where $x_k \in M_0$):

$$\begin{aligned} & \|x_1 \dots \delta(x_k) \dots x_p - x_{1,\epsilon} \dots \delta(x_{k,\epsilon}) \dots x_{p,\epsilon}\|_2 \leq \|x_{1,\epsilon} \dots \delta(x_{k,\epsilon}) \dots x_{p,\epsilon} - x_{1,\epsilon} \dots \delta(x_k) \dots x_{p,\epsilon}\|_2 \\ & + \|x_{1,\epsilon} \dots (x_{k-1} - x_{k-1,\epsilon}) \delta(x_k) \dots x_{p,\epsilon}\|_2 + \dots + \|(x_1 - x_{1,\epsilon}) \dots x_{k-1} \delta(x_k) \dots x_{p,\epsilon}\|_2 \\ & + \|x_1 \dots x_{k-1} \delta(x_k) (x_{k+1} - x_{k+1,\epsilon}) \dots x_{p,\epsilon}\|_2 + \dots + \|x_1 \dots x_{k-1} \delta(x_k) x_{k+1} \dots (x_p - x_{p,\epsilon})\|_2 \\ & \leq pR^{p-1}\epsilon/2Mp^2R^{p-1}. \end{aligned}$$

Thus we obtain a bound in $\epsilon/2M$ for each term in the sum of products, with M terms. This concludes $\|\delta(x_\epsilon - y_\epsilon)\|_2 \leq \epsilon/2$.

For the corresponding statement for $\delta^*\delta$, one first consider a projection $P = P_{i_1, \dots, i_n}$ on the space of terms generated by $a_1 \xi_{i_1} a_2 \dots \xi_{i_n} a_{n+1}$, $a_i \in L^2(M_0)$ $\tau(a_i) = 0$ if $i \neq 1, n+1$, ξ_i an orthonormal basis of the Brownian motion space starting with $\xi_0 = 1$, $i_j > 0$. Then we show $D(\Delta)$ stable by P with a commutation in the spirit of corollary 17. The conclusion follows easily. \square

2.1.4. *Summary of results under assumption 0.* We summarize the easy results obtained at this stage :

Lemma 19. *With those assumptions $0'$, $\bar{\delta}$ and Δ satisfy the stability of filtration properties and $\Gamma_0(\omega = 0)$ and also b' , c' (i.e. assumptions $a), b), b'), c), c'$ and $d)$ of $\Gamma_1(\omega = 0, C)$.*

■

2.1.5. *Applications of almost coassociativity to almost commutation with Δ .* Before discussing coassociativity, let us discuss a few notations.

Let us define

$$B_{vN}^1(M_s) = \{x \in M_s \cap D(\bar{\delta}) \mid \bar{\delta}(x) \in (M_s \bar{\otimes} M_s)^N\},$$

where we see the von Neumann tensor products continuously embedded in $L^2(M_s) \otimes L^2(M_s)$.

Let us also fix some similar notations, let us write $\overline{\delta_i^{(j,n-j)}} = \overline{1^{\otimes(j-1)} \otimes \delta_i \otimes 1^{\otimes(n-j)}} : L^2(M)^{\otimes n} \rightarrow L^2(M)^{\otimes(n+1)}$ and the variants before closure with value in algebraic tensor products of L^p spaces (p given by 0_p), and the analogue $L^r \rightarrow L^s$ variant $\delta_i^{(j,n-j);r,s}$ using lemma 15 (maybe with an O_i if an opposite algebra is used, i giving the position, e.g. $\delta_i^{O_2(1,1);r,s} : L^r(M \otimes M) \rightarrow L^s(M \otimes M^{op} \otimes M)$, $\delta_i^{O_3(1,1);r,s} : L^r(M \otimes M) \rightarrow L^s(M \otimes M \otimes M^{op})$).

We now introduce a notion of almost coassociativity to motivate the next definition.

In the spirit of a notation we will often use and introduce more generally later, for $U \in L^2(M \otimes M^{op})$, $C \in M \otimes M \otimes M^{op}$ we write $U \# C \in L^2(M \otimes M \otimes M^{op})$ the extension in the usual way given in the case $U = a \otimes b$ $C = d \otimes e \otimes f$ by $U \# C = ad \otimes e \otimes bf$ (multiplication in M^{op} of b and f corresponding thus to the expected fb in M). Said otherwise since for $C = 1$ the meaning of the isometric $\cdot \# 1$ is clear, $U \# C = (U \# 1)C$ with the action of $M \otimes M \otimes M^{op}$ on its L^2 space. We may also use the same notation for $U \in L^2(M \otimes M)$ so that $1 \otimes O(U) \# C = 1 \otimes 1 \otimes O(U \# C)$.

Definition 20. Consider $\partial = (\partial_1, \dots, \partial_N)$ and $\tilde{\partial} = (\tilde{\partial}_{N+1}, \dots, \tilde{\partial}_M)$ satisfying assumption $0'$. We say ∂ is **almost coassociative** with respect to $\tilde{\partial}$ with defect $C = (C_{ij}^k, C_{ji}^k)$, ($j = 1 \dots N$, $i, k = N+1 \dots M$ $C \in (M_0 \bar{\otimes} M_0 \bar{\otimes} M_0^{op})^{2NM^2}$) if for any $x \in D(\partial) \cap D(\tilde{\partial})$ such that for all

$i = 1 \dots N$ $\partial_i(x) \in D(\overline{\tilde{\partial} \otimes 1}) \cap D(\overline{1 \otimes \tilde{\partial}})$ and for all $i = 1 \dots M$ $\tilde{\partial}_i(x) \in D(\overline{\partial \otimes 1}) \cap D(\overline{1 \otimes \partial})$ then the following relations hold :

$$\begin{aligned} (\overline{\partial_j \otimes 1}) \circ \tilde{\partial}_i(x) - (\overline{1 \otimes \tilde{\partial}_i}) \circ \partial_j(x) &= \sum_{k=1}^M \tilde{\partial}_k(x) \# C_{j,i}^k =: \tilde{\partial}(x) \# C_{j,i}, \\ (\overline{\tilde{\partial}_i \otimes 1}) \circ \partial_j(x) - (\overline{1 \otimes \partial_j}) \circ \tilde{\partial}_i(x) &= \sum_{k=1}^M \tilde{\partial}_k(x) \# C_{i,j}^k =: \tilde{\partial}(x) \# C_{i,j}. \end{aligned}$$

If $\partial = \tilde{\partial}$, we say it is almost coassociative.

Note that if we think of the left hand sides above as coassociators in analogy with commutators, almost coassociativity (for a single ∂) is an analog of being a Lie algebra, the coassociator stays a linear combination with coefficient $M \otimes M \otimes M^{op}$. For our purposes, supplementary regularity properties of C will be crucial.

In order to explain this regularity properties of C , we want to show how almost coassociativity enables us to make almost commute δ and $\tilde{\delta}^* \tilde{\delta}$, or more generally δ and $\delta^{(1)*} \delta^{(2)}$. This will motivate the following ad-hoc definitions of Sobolev-like spaces, (almost) minimally chosen to enable our computations (at a first "algebraic" level). Since we don't have a unique choice in terms of L^ρ -regularity and almost all consistent one bellow would work in our applications, we thus provide variants (and emphasize those we will use). These variants depend on constraints that will be justified later either for enabling computations or to see B^2, B^3 are algebras, we thus give names to each set of useful constraints.

Thus consider $\delta_{(2)}$ (resp. $\delta_{(1)}, \delta$) satisfying assumption $0'_{\pi_{(2)}}$ (resp. $0^a_{\pi_{(1)}}, 0^{a'}_{p,q}$, recall, this assumes $p, \pi_{(i)} \geq 2$). We call

Def $_{\mathbf{p}, \pi_{(2)}}(\rho) : p, \pi_{(2)} \geq 2$ and $\rho \in (1, p]$.

Alg $_{\mathbf{p}, \pi_{(2)}}(\rho) : p, \pi_{(2)} \geq 2$ and $\rho \in (1, \tilde{\pi}]$ where $1/\tilde{\pi} = 1/p + 1/\pi_{(2)}$.

Consider ρ, ρ', κ such that $Def_{p, \pi_{(2)}}(\rho), Def_{\pi_{(1)}, \pi_{(2)}}(\rho'), Def_{p, \pi_{(1)}}(\kappa)$ holds so that $\delta^{O; \pi_{(2)}, \rho}, \delta^{O; \pi_{(2)}, \rho'}$, $\delta^{O; (2,0); \pi_{(1)}, \kappa}, \delta^{O; (1,1); \pi_{(1)}, \kappa}$ are defined by lemma 15. Then we can define :

$$\begin{aligned} B^2_{\pi_{(2)}, \rho'}(M_s, \delta_{(1)}, \delta_{(2)}) &= \{x \in M_s \cap D(\delta_{(2)}) \cap D(\delta_{(1)}) \mid \delta_{(2)i}(x) \in D(\delta^{O; \pi_{(2)}, \rho'}) \otimes_{alg} D(\delta^{O; \pi_{(2)}, \rho'})\}, \\ B^{2,O}_{\pi_{(2)}, \rho}(M_s, \delta, \delta_{(2)}) &= \{x \in M_s \cap D(\delta_{(2)}) \cap D(\delta) \mid \delta_{(2)i}(x) \in D(\delta^{O; \pi_{(2)}, \rho}) \otimes_{alg} D(\delta^{O; \pi_{(2)}, \rho})\}, \\ B^{2,a}_{\pi_{(1)}, \kappa}(M_s, \delta, \delta_{(1)}) &= \{x \in M_s \cap D(\delta_{(1)}) \cap D(\delta) \mid \delta_{(1)i}(x) \in D(\delta^{O; (2,0); \pi_{(1)}, \kappa}) \cap D(\delta^{O; (1,1); \pi_{(1)}, \kappa})\}. \end{aligned}$$

We call $\Pi = (p, q, \pi_{(1)}, \pi_{(2)})$, $R = (\rho, \rho', \kappa, \kappa', \sigma)$ for which we define the following set of conditions :

Def $_{\Pi}^3(\mathbf{R})$: $p, q, \pi_{(1)}, \pi_{(2)} \geq 2, \rho, \kappa \in (1, p], \rho', \kappa' \in (1, \pi_{(1)}]$ and $1/\rho' \leq \min(1 - 1/q, 1 - 1/p), \rho \geq 2; \sigma \in (1, \min(p, \pi_{(1)}))$.

Alg $_{\Pi}^3(\mathbf{R})$: $Def_{\Pi}^3(R), Alg_{p, \pi_{(2)}}(\rho), Alg_{\pi_{(1)}, \pi_{(2)}}(\rho'), Alg_{p, \pi_{(1)}}(\kappa), Alg_{\pi_{(1)}, p}(\kappa'),$
 $1/p + 1/\pi_{(2)} \leq 1/2$ and $1/\sigma \geq \max(1/\rho + 1/\pi_{(1)}, 1/\rho' + 1/p, 1/\kappa, 1/\kappa')$.

CoAss $_{(\Pi)}^3(\rho, \rho', \varrho_1, \sigma, \varpi_1)$: $Def_{\Pi}^3(\rho, \rho', p, \pi_1, \sigma), \varpi_1 \in (1, \pi_{(1)}]; \varrho_1 \geq 2, 1/\sigma + 1/\pi_{(2)} \leq 1$
and $1/\varpi := \max(1/\pi_{(2)} + 1/q, 1/\pi_{(2)} + 1/p, 1/\rho) \leq 1 - 1/\rho',$
and $1/\varpi' := \max(1/\pi_{(2)} + 1/\varrho_1, 1/\varpi_1 + 1/\pi_{(2)}, 1/\rho') \leq 1 - 1/\rho.$

Comp $_{\Pi}^3(\mathbf{R}, \varrho_1, \varsigma_1, \varpi_1)$: $\varrho_1 \geq 2, Alg_{\Pi}^3(R), \varpi_{(1)} \in (1, \pi_{(1)}]; 1/\varpi_{(1)} + 1/\pi_{(2)} + 1/\rho \leq 1$
and $1/\tilde{\sigma} := \max(1/\rho + 1/\rho', 1/\sigma + 1/\pi_{(2)}, 1/\rho + 1/\pi_{(2)} + 1/\varrho_1, 2/\pi_{(2)} + 1/\varsigma_1) < 1,$
and $1/\tilde{\rho} := \max(1/\rho' + 1/\pi_{(2)}, 2/\pi_{(2)} + 1/\varrho_1) \leq \min(1 - 1/q, 1 - 1/p).$

Remark 21. Note that $Comp_{\Pi}^3(R, \varrho_1, \varsigma_1, \varpi_1) \Rightarrow CoAss_{(\Pi)}^3(\rho, \rho', \varrho_1, \sigma, \varpi_1)$ since $\max(1/\varpi + 1/\rho', 1/\varpi' + 1/\rho) \leq \max(1/\varpi_{(1)} + 1/\pi_{(2)} + 1/\rho, 1/\tilde{\sigma}, 1/\tilde{\rho} + 1/q, 1/\tilde{\rho} + 1/p)$ (the only added conditions are those on ϖ_1). Let us give our two main examples $\Pi_{\infty} = (\infty, 2, \infty, \infty)$ (or a fortiori $(\infty, q, \infty, \infty)$ $q \geq 2$), $R_{\infty, \epsilon} = (2 + \epsilon, 2 + \epsilon, 1 + \epsilon/2, 1 + \epsilon/2, 1 + \epsilon/2)$ for any $\epsilon > 0$ then we have $Comp_{\Pi_{\infty}}^3(R_{\infty, \epsilon}, 2, 2, 2)$ (with then $1/\tilde{\sigma} = 1/2 + 1/(2 + \epsilon), \tilde{\rho} = 2$). If we want to comment, here we assume almost nothing on $q = 2$, but a lot on $p, \pi'_{(i)}$ s. We can also assume less on them with a price to pay on q . Let $\Pi_{4, \epsilon} = (4 + \epsilon, 4, 4 + \epsilon, 4 + \epsilon)$, $R_{4, \epsilon} = (2 + \epsilon/2, 2 + \epsilon/2, 4/3 + \epsilon/3, 4/3 + \epsilon/3, 4/3 + \epsilon/3)$ then $Comp_{\Pi_{4, \epsilon}}^3(R_{4, \epsilon}, 4 + \epsilon + \eta, 2 + \epsilon/2 + \eta, 4)$ holds for any $\epsilon > 0, \eta \geq 0$ (with $\tilde{\sigma} = 1 + \epsilon/4, \tilde{\rho} = 4/3 + \epsilon/3$). In our applications in section 4, either the first case will work (free difference quotient) or both but with more or less work (q-gaussians).

We now assume $Def_{\Pi}^3(R)$ enabling to define $\delta^{O; \rho', \sigma}$ and $\delta_{(1)}^{2, \sigma}$ via lemma 15. Moreover we can define in the same spirit $(\delta \oplus \delta_{(1)})^{O; \pi_{(2)}, \rho \oplus \rho'} : L^{\pi_{(2)}} \rightarrow (L^{\rho}(M \otimes M^{op}))^N \oplus (L^{\rho'}(M \otimes M))^N$ a simultaneous closure of $1 \otimes O\partial \oplus \partial_{(1)}$ if $\partial \oplus \partial_{(1)}$ (domain by definition the intersection of domains) satisfy assumption $0'$ We write $\mathcal{O} = 1 \otimes O^{-1} \circ I$ with $I : L^{\rho} \rightarrow L^2$ the inclusion. (Here $\otimes_a = \otimes_{alg}$ is also algebraic tensor product)

$$\begin{aligned} B_{\Pi, R}^3(M_s, \delta, \delta_{(1)}, \delta_{(2)}) \\ = \{x \in B_{\pi_{(2)}, \rho}^{2, O}(M_s, \delta, \delta_{(2)}) \cap B_{\pi_{(2)}, \rho'}^2(M_s, \delta_{(1)}, \delta_{(2)}) \cap B_{\pi_{(1)}, \kappa}^{2, a}(M_s, \delta, \delta_{(1)}) \cap B_{p, \kappa'}^{2, a}(M_s, \delta_{(1)}, \delta) \mid \\ \delta_{(2)i}(x) \in D((\delta \oplus \delta_{(1)})^{O; \pi_{(2)}, \rho \oplus \rho'}) \otimes_a D((\delta \oplus \delta_{(1)})^{O; \pi_{(2)}, \rho \oplus \rho'}) \\ (\mathcal{O}\delta_j^{O; \pi_{(2)}, \rho}) \otimes 1\delta_{(2)i}(x) \in D(\delta_{(1)}^{O_2; (2, 0); 2, \sigma}) \otimes_a D(\delta_{(1)}^{\pi_{(2)}, \rho'}), \delta_{(1)j}^{\pi_{(2)}, \rho'} \otimes 1\delta_{(2)i}(x) \in D(\delta^{O; (1, 1); \rho', \sigma}) \otimes_a D(\delta^{\pi_{(2)}, \rho}) \\ 1 \otimes (\mathcal{O}\delta_j^{O; \pi_{(2)}, \rho})\delta_{(2)i}(x) \in D(\delta_{(1)}^{\pi_{(2)}, \rho'}) \otimes_a D(\delta_{(1)}^{O_3; (1, 1); 2, \sigma}), 1 \otimes \delta_{(1)j}^{\pi_{(2)}, \rho'}\delta_{(2)i}(x) \in D(\delta^{\pi_{(2)}, \rho}) \otimes_a D(\delta^{O, (2, 0); \rho', \sigma}) \\ \delta_i(x) \in L^2(M_s) \otimes D(\overline{\delta_{(2)}}) \cap D(\overline{\delta_{(2)}}) \otimes L^2(M_s)\} \end{aligned}$$

Thus we first sketch the proof of the following lemma (that will be conveniently used with lemma 18) :

Lemma 22. *Let us assume $\delta_{(2)}$ (resp. $\delta_{(1)}, \delta$) satisfy assumption $0'_{\pi_{(2)}}$ (resp. $0_{\pi_{(1)}}^a, 0_{p, q}^{a'}$) If we assume $Alg_{p, \pi_{(2)}}(\rho)$ (resp. $Alg_{\pi_{(1)}, \pi_{(2)}}(\rho'), Alg_{p, \pi_{(1)}}(\kappa)$) then $B_{\pi_{(2)}, \rho}^{2, O}(M_s, \delta, \delta_{(2)})$, (resp $B_{\pi_{(2)}, \rho'}^2(M_s, \delta_{(1)}, \delta_{(2)})$, $B_{\pi_{(2)}, \rho}^{2, a}(M_s, \delta, \delta_{(1)})$) are $*$ -subalgebras of M_s .*

If we assume $Alg_{\Pi}^3(R)$ for $\Pi = (p, q, \pi_{(1)}, \pi_{(2)})$, $R = (\rho, \rho', \kappa, \kappa', \sigma)$ with $\delta, \delta_{(1)}, \delta_{(2)}$ satisfying $0'_{p,q}, 0_{\pi_1}^a$ and 0_{π_2} respectively, $\partial \oplus \partial_{(1)}$ satisfying assumption $0'$, then $B_{\Pi,R}^3(M_s, \delta, \delta_{(1)}, \delta_{(2)})$ is a $*$ -subalgebra of M_s .

Proof. Stability by adjoint comes from the realness of all our derivations and symmetry of the defining conditions of B^2, B^3 . Of course, stability by product comes from derivation relations (coming from lemma 15 for extensions thus using mainly also Hölder inequality), e.g. for $x, y \in B_{\pi_{(2)},\rho}^{2,O}(M_s, \delta, \delta_{(2)})$ (with for $c \in L^{\pi_2}(M)$, $b \in L^p(M^{op})$ $m(b \otimes c) = O(c)b \in L^p(M^{op})$ and its extension to algebraic tensor products):

$$(\delta_i^{O;\pi_{(2)},\rho} \otimes 1)\delta_{(2)j}(xy) = (\delta_i^{O;\pi_{(2)},\rho} \otimes 1)\delta_{(2)j}(x)y + x\delta_i^{O;\pi_{(2)},\rho} \otimes 1\delta_{(2)j}(y) + (1 \otimes m \otimes 1)((1 \otimes O)\delta_i(x) \otimes \delta_{(2)j}(y)).$$

Here lemma 15 (or a variant for $\delta \otimes 1$) is used in the third case with notation $r = \infty$ $s = p$ and with $r'' = r' = \pi_2$ ($\delta_i^{O;2,\rho}$ coinciding with $\delta_i^{O;\pi_{(2)},\rho}$ on the latter domain) $s'' = s' = \rho$ and this is possible because the only non trivial relation to check is $1/s'' \geq 1/s + 1/r'$, which is exactly what is given in $Alg_{p,\pi_{(2)}}(\rho)$.

Analogously for $x, y \in B_{\Pi,R}^3(M_s, \delta, \delta_{(1)}, \delta_{(2)})$ we have (with various m) e.g. :

$$\begin{aligned} & (\delta_k^{O;(1,1);\rho',\sigma} \otimes 1)(\delta_{(1)i}^{\pi_2,\rho'} \otimes 1)\delta_{(2)j}(xy) \\ &= (\delta_k^{O;(1,1);\rho',\sigma} \otimes 1)(\delta_{(1)i}^{\pi_2,\rho'} \otimes 1)\delta_{(2)j}(x)y + x(\delta_k^{O;(1,1);\rho',\sigma} \otimes 1)(\delta_{(1)i}^{\pi_2,\rho'} \otimes 1)\delta_{(2)j}(y) \\ &+ (1 \otimes m \otimes 1 \otimes 1)((1 \otimes O\delta_k(x) \otimes ((\delta_{(1)i}^{\pi_2,\rho'} \otimes 1)\delta_{(2)j}(y))) \\ &+ (1 \otimes 1 \otimes m \otimes 1)(\delta_k^{O;(1,1),\pi_{(1)},\kappa} \delta_{(1)i}(x) \otimes \delta_{(2)j}(y)). \end{aligned}$$

As before what is key (for the third term) is to have $1/\sigma \geq 1/p + 1/\rho'$. We also use on the fourth term that $\delta_k^{O;(1,1);\rho',\sigma}$ coincide with $\delta_k^{O;(1,1),\pi_{(1)},\kappa}$ on the domain of the latter since $\pi_{(1)} \geq \rho', \kappa \geq \sigma$, the last product is in L^σ , since a consequence of other assumptions give $1/\sigma \geq 1/\pi_{(1)} + 1/\pi_{(2)}$. Finally, note that the condition $1/p + 1/\pi_{(2)} \leq 1/2$ is assumed to make the last condition in the definition of B^3 involving $\delta, \delta_{(2)}$ multiplicative. \square

Remark 23. Consider ∂ almost coassociative with respect to $\tilde{\partial}$ with defect C , and respectively satisfying assumption $0_p^{a'}$ and 0_r^a thus giving $\delta, \tilde{\delta}$. Assume moreover $Alg_{p,r}(\rho_1), Alg_{r,p}(\rho_2), \rho_1, \rho_2 \geq 2$ and $B_{p,r,\rho_1,\rho_2}^{2,sym}(M_0, \delta, \tilde{\delta}) = B_{r,\rho_1}^{2,a}(M_0, \delta, \tilde{\delta}) \cap B_{p,\rho_2}^{2,a}(M_0, \tilde{\delta}, \delta)$ is a core for $\partial \oplus \tilde{\partial}$. Then, since both sides of almost coassociativity relations are derivations, they are checked by δ and $\tilde{\delta}$ on the algebra $B_{p,r,\rho_1,\rho_2}^{2,sym}(M_0, \delta, \tilde{\delta}) * \mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle$, which is a core for $\delta \oplus \tilde{\delta}$, by lemma 18. A small duality argument (to move the second derivative in the right hand side of a scalar product in the spirit of the next lemma) thus shows δ is also almost coassociative with respect to $\tilde{\delta}$ with same defect C .

We can now state the "algebraic" version of our almost commutation result coming from almost coassociativity.

Proposition 24. Assume $Comp_{\Pi}^3(R, \varrho_1, \varsigma_1, \varpi_1)$ for $\Pi = (p, q, \pi_{(1)}, \pi_{(2)})$, $R = (\rho, \rho', \kappa, \kappa', \sigma)$ with $\delta, \delta_{(1)}, \delta \oplus \delta_{(1)}, \delta_{(2)}$ satisfying $0_{p,q}^{a\delta_{(1)},\varpi_1}, 0_{\pi_1,\varrho_1}^a, 0'$ and $0'_{\pi_2}$ respectively, and $\forall i, \delta_{(1)i}^* 1 \otimes 1 \in D(\delta^{O;\varrho_1,\varsigma_1})$. We also assume $\delta_{(1)}, \delta_{(2)}$ almost coassociative with respect to δ with defects

$C = (C_{i,j(1)}^k, C_{j(1),i}^k), D = (D_{i,j(2)}^k, D_{j(2),i}^k) \in (M_0 \overline{\otimes} M_0 \overline{\otimes} M_0^{op})^{2N^3}$ respectively. We also assume $Alg_{p,r}(\rho_1), Alg_{r,p}(\rho_2), \rho_1, \rho_2 \geq 2$ and $B_{p,r,\rho_1,\rho_2}^{2,sym}(M_0, \delta, \delta_{(1)})$ is a core for $\overline{\partial \oplus \partial_{(1)}}$.

Let $x \in B_{\Pi,R}^3(M_s, \delta, \delta_{(1)}, \delta_{(2)})$ then $x \in D(\delta_{(1)}^* \delta_{(2)})$ and $\delta_{(1)}^* \delta_{(2)}(x) \in D(\delta^{O;\tilde{\rho},\tilde{\sigma}})$. Moreover, we have, for any $U \in (D(\overline{\delta_{(1)}}) \cap M) \otimes_{alg} (D(\overline{\delta_{(1)}}) \cap M)$, the relation :

$$\begin{aligned} \langle \delta_i^{O;\tilde{\rho},\tilde{\sigma}}(\delta_{(1)j}^* \delta_{(2)j}(x)), (1 \otimes O)U \rangle &= \langle ((1 \otimes O)\delta_{(2)j}(x)) \# \delta_i^{O;\varrho_1,\varsigma_1} \delta_{(1)j}^*(1 \otimes 1), (1 \otimes O)(U) \rangle \\ &+ \langle (1 \otimes \overline{\delta_{(2)j}}) \circ \delta_i(x), (1 \otimes \overline{\delta_{(1)j}})(U) \rangle + \langle (\overline{\delta_{(2)j}} \otimes 1) \circ \delta_i(x), (\overline{\delta_{(1)j}} \otimes 1)(U) \rangle \\ &+ \langle \sum_{k=1}^N \delta_k(x) \# D_{i,j(2)}^k, (1 \otimes \overline{\delta_{(1)j}})(U) \rangle - \sum_{k=1}^N \langle 1 \otimes [m \circ 1 \otimes \tau \otimes 1] (((1 \otimes \overline{\delta_{(2)j}}) \circ \delta_k(x)) \#_1 C_{i,j(1)}^k), U \rangle \\ &- \langle \sum_{k=1}^N \delta_k(x) \# D_{j(2),i}^k, (\overline{\delta_{(1)j}} \otimes 1)(U) \rangle + \sum_{k=1}^N \langle [m \circ 1 \otimes \tau \otimes 1] \otimes 1 (((\overline{\delta_{(2)j}} \otimes 1)(\delta_k(x)) \#_2 C_{j(1),i}^k), U) \rangle \\ &- \sum_{k=1}^N \langle [m \circ 1 \otimes \tau \otimes 1] \otimes 1 \left(\left(\sum_{l=1}^N \delta_l(x) \# D_{j(2),k}^l \right) \#_2 C_{j(1),i}^k \right), U \rangle \\ &- \sum_{k=1}^N \langle 1 \otimes [m \circ 1 \otimes \tau \otimes 1] \left(\left(\sum_{l=1}^N \delta_l(x) \# D_{k,j(2)}^l \right) \#_1 C_{i,j(1)}^k \right), U \rangle \end{aligned}$$

We can see right now what we will need assuming (beyond almost coassociativity) and proving to go from there to Γ_1, f : first we will have to assume properties of the defect to get the four last lines of the form $\langle \mathcal{H}\delta, U \rangle$ and, for this, also prove or assume boundedness of maps of the form $1 \otimes \tau \delta_{(i)}$. Second, we will have to get the result above on a better domain and solve domain issues to get also e). This will be the main topics of the next subsection.

We need several preliminaries before the proof. Before using almost coassociativity, we now want to see the way we will apply it on B^3 or B^2 . Here we will write $J_f(a \otimes b \otimes c) = c^* \otimes b^* \otimes a^*$, $J(a \otimes b \otimes c) = a^* \otimes b^* \otimes c^*$ and natural extensions to various tensor products.

Lemma 25. Assume $CoAss_{(\Pi)}^3(\rho, \rho', \varrho_1, \sigma, \varpi_1)$ with $\delta, \delta_{(1)}$, satisfying $0_{p,q}^{a\delta_{(1)},\varpi_1}, 0_{\pi_1,\varrho_1}^a$ respectively, $\partial \oplus \partial_{(1)}$ satisfying assumption O' . We also assume $\delta_{(1)}$ almost coassociative with respect to δ with defect $C = (C_{i,j(1)}^k, C_{j(1),i}^k) \in (M_0 \overline{\otimes} M_0 \overline{\otimes} M_0^{op})^{2N^3}$. We write $\tilde{\delta}_i = \delta_i \# I_i^{-1} : D(\delta) \rightarrow L^2(M \otimes M)$ the derivation given by ϖ_1 -regular reducibility. Recall $O : M \rightarrow M^{op}$ the identity map, \mathcal{O} defined before B^3 .

Finally, for (1) and (3), we assume $Alg_{p,r}(\rho_1), Alg_{r,p}(\rho_2), \rho_1, \rho_2 \geq 2$ and $B_{p,r,\rho_1,\rho_2}^{2,sym}(M_0, \delta, \delta_{(1)})$ is a core for $\overline{\partial \oplus \partial_{(1)}}$.

- (1) Then, for any $U \in D((\delta \oplus \delta_{(1)})^{O;\pi(2),\rho \oplus \rho'}) \otimes_{alg} D((\delta \oplus \delta_{(1)})^{O;\pi(2),\rho \oplus \rho'})$, we have, for any $b, c \in D(\delta_{(1)}), a, b \in D(\delta)$ b seen in M^{op} :

²Recall we defined and assumed $1/\tilde{\sigma} := \max(1/\sigma + 1/\pi_{(2)}, 1/\rho + 1/\pi_{(2)} + 1/\varrho_1, 2/\pi_{(2)} + 1/\varsigma_1) < 1$, $1/\tilde{\rho} := \max(1/\rho' + 1/\pi_{(2)}, 2/\pi_{(2)} + 1/\varrho_1) \leq \min(1 - 1/q, 1 - 1/p)$

$$\begin{aligned}
& \tau(m \circ (1 \otimes \tau \otimes 1)(\tilde{\delta}_i^{*,1} \otimes 1 \otimes 1)[a \otimes ((O^{-1}(b) \otimes c \otimes 1)J_f(\delta_{(1),j}^{\pi(2),\rho'} \otimes 1(U))))]) \\
& - \tau \otimes \tau((1 \otimes m \circ (1 \otimes \delta_{(1),j}^{*,1}))[\(((a \otimes O^{-1}(b))\#I_j^{-1*})\#1 \otimes 1 \otimes 1)J(\delta_i^{O;\pi(2),\rho} \otimes 1(U)) \otimes c]) \\
& = \sum_{k=1}^N \langle 1 \otimes m(1^{\otimes 2} \otimes \tau \otimes 1)[((1 \otimes O \otimes 1) \circ (\mathcal{O}\delta_k^{O;\pi(2),\rho} \# C_{i,j(1)}^k)) \otimes 1(U)(1^{\otimes 2} \otimes c^* \otimes 1)], (a \otimes b)\#I_j^{-1*} \rangle.
\end{aligned}$$

(2) For any $V \in D(\delta^{O;(1,1);\rho',\sigma}) \otimes_{alg} D(\delta^{O;\pi(2),\rho})$, $W; \mathcal{O} \otimes 1(W) \in D(\delta_{(1)}^{O_2;(2,0);2,\sigma}) \otimes_{alg} D(\delta_{(1)}^{\pi(2),\rho'})$, for any $b, c \in D(\delta_{(1)})$, $a, b \in D(\delta)$ b seen in M^{op} :

$$\begin{aligned}
& \tau(m \circ (1 \otimes \tau \otimes 1)(\tilde{\delta}_i^{*,1} \otimes 1 \otimes 1)[a \otimes ((O^{-1}(b) \otimes c \otimes 1)J_f(V))]) \\
& = \langle (1 \otimes (m \circ (1 \otimes \tau \otimes 1)))[(\tilde{\delta}_i^{O;(1,1);\rho',\sigma} \otimes 1(V))(1 \otimes 1 \otimes c^* \otimes 1)], a \otimes b \rangle \\
& \tau \otimes \tau((1 \otimes m \circ (1 \otimes \delta_{(1),j}^{*,1}))[\(((a \otimes O^{-1}(b))\#I_j^{-1*})\#1 \otimes 1 \otimes 1)J(W) \otimes c]) \\
& = \langle (1 \otimes (m \circ (1 \otimes \tau \otimes 1)))[(\delta_{(1),j}^{O_2;(2,0);2,\sigma} \mathcal{O} \otimes 1(W))(1 \otimes 1 \otimes c^* \otimes 1)], (a \otimes b)\#I_j^{-1*} \rangle
\end{aligned}$$

(3) For any $U \in D((\delta \oplus \delta_{(1)})^{O;\pi(2),\rho \oplus \rho'}) \otimes_{alg} D((\delta \oplus \delta_{(1)})^{O;\pi(2),\rho \oplus \rho'})$ such that $\delta_{(1),j}^{\pi(2),\rho'} \otimes 1(U) \in D(\delta^{O;(1,1);\rho',\sigma}) \otimes_{alg} D(\delta^{\pi(2),\rho})$ and $\mathcal{O}\delta_i^{O;\pi(2),\rho} \otimes 1(U) \in D(\delta_{(1)}^{O_2;(2,0);2,\sigma}) \otimes_{alg} D(\delta_{(1)}^{\pi(2),\rho'})$, for any $c \in M$, we have :

$$\begin{aligned}
& (1 \otimes (m \circ (1 \otimes \tau \otimes 1)))[\{(\delta_i^{O;(1,1);\rho',\sigma} \otimes 1(\delta_{(1),j}^{\pi(2),\rho'} \otimes 1(U))) \\
& - (\delta_{(1),j}^{O_2;(2,0);2,\sigma} \otimes 1(\mathcal{O}\delta_i^{O;\pi(2),\rho} \otimes 1(U)))\}(1 \otimes 1 \otimes c^* \otimes 1)] \\
& = \sum_{k=1}^N 1 \otimes m(1 \otimes 1 \otimes \tau \otimes 1)[((1 \otimes O \otimes 1)((\mathcal{O}\delta_k^{O;\pi(2),\rho} \# C_{i,j(1)}^k) \otimes 1)(U)(1 \otimes 1 \otimes c^* \otimes 1)]
\end{aligned}$$

Proof. For (1), first note that terms involving $\tilde{\delta}_i^{*,1}$, and $\delta_{(1),i}^{*,1}$ are defined and in L^ϖ resp. $L^{\varpi'}$ via lemma 16 (ϖ, ϖ' defined in $CoAss^3$). Second, without loss of generality $U = d \otimes e$. Then since the terms $\delta_{(1)}(u), \delta(u), u = d, e$ involved are continuous in $L^{\rho'}$ or L^ρ , we can assume $d, e \in D(\delta) \cap D(\delta_{(1)})$ and even by the core property (since terms in $\delta_{(1)}(u), \delta(u), u = d, e$ involved are continuous in L^2 once $u \in M$, this is nothing but what we already noticed with the conditions on ϖ, ϖ' changed to $\pi_{(2)} = \infty$ are satisfied if ρ, ρ' also become 2), satisfying the conditions for almost coassociativity. Then the computation reduces to it via formulas in the spirit of (2) we now detail. As in (1), for (2) (using here $1/\sigma + 1/\pi_{(2)} \leq 1$), we are reduced to the case $V = d \otimes e \otimes f$, $d, f \in D(\delta)$ $e \in M$ (likewise for $W = d \otimes e \otimes f$, $e, f \in D(\delta_{(1)})$

$d \in M$), the formula then becomes :

$$\begin{aligned}
& \tau(m \circ (1 \otimes \tau \otimes 1)(\tilde{\delta}_i^{*1} \otimes 1 \otimes 1)[a \otimes ((O^{-1}(b) \otimes c \otimes 1)J_f(d \otimes e \otimes f))]) \\
&= \tau(\delta_i^*(a \otimes O^{-1}(b)f^*)d^*)\tau(ce^*) = \langle (1 \otimes O)\delta_i(d) \otimes e, a \otimes O(f^*)b \otimes c \rangle \\
&= \langle (1 \otimes (m \circ (1 \otimes \tau \otimes 1)))[(\tilde{\delta}_i^{O;(1,1);\rho',\sigma} \otimes 1(d \otimes e \otimes f))(1 \otimes 1 \otimes c^* \otimes 1)], a \otimes b \rangle \\
&= \tau \otimes \tau((1 \otimes m \circ (1 \otimes \delta_{(1),j}^{*1}))[(a \otimes 1 \otimes O^{-1}(b))J(d \otimes e \otimes f) \otimes c]) \\
&= \tau(ad^*)\tau(e^*\delta_{(1),j}^*(O^{-1}(b)f^* \otimes c)) = \langle d \otimes (O \otimes 1)\delta_{(1),j}(e), a \otimes O(f^*)b \otimes c \rangle \\
&= \langle (1 \otimes (m \circ (1 \otimes \tau \otimes 1)))[(\delta_{(1),j}^{O_1;(2,0);2,\sigma} \mathcal{O} \otimes 1(d \otimes e \otimes f))(1 \otimes 1 \otimes c^* \otimes 1)], a \otimes b \rangle
\end{aligned}$$

(3) only gathers (1) and (2), extend values of c by density, and multiply the result by I_i . \square

The much easier next lemma was essentially used in rmk 23 (since $\rho \geq 2$ this is a direct application of the definition), we state it for reference :

Lemma 26. *Consider $\delta, \delta_{(2)}$, satisfying $0_p^a, 0_{\pi_2}$ respectively. We also assume $\delta_{(2)}$ almost coassociative with respect to δ with defect $D = (D_{i,j(2)}^k, D_{j(2),i}^k) \in (M_0 \overline{\otimes} M_0 \overline{\otimes} M_0^{op})^{2N^3}$.*

Then consider $\rho \in [2, p]$, then for any $x \in B_{\pi(2),\rho}^{2,O}(M_0, \delta, \delta_{(2)})$ such that $\delta_i(x) \in L^2(M_s) \otimes D(\overline{\delta_{(2)}}) \cap D(\overline{\delta_{(2)}}) \otimes L^2(M_s)$, then

$$\begin{aligned}
(\mathcal{O}\delta_i^{O;\pi(2),\rho} \otimes 1) \circ \delta_{(2)j}(x) - (1 \otimes \overline{\delta_{(2)j}}) \circ \delta_i(x) &= \sum_{k=1}^M \delta_k(x) \# D_{i,j(2)}^k, \\
(\overline{\delta_{(2)j}} \otimes 1) \circ \delta_i(x) - (1 \otimes \mathcal{O}\delta_i^{O;\pi(2),\rho}) \circ \delta_{(2)j}(x) &= \sum_{k=1}^M \delta_k(x) \# D_{j(2),i}^k.
\end{aligned}$$

■

We will need an obvious analog of lemma 16 for $(\delta_{(1)} \otimes 1)^{*1}$ on $L^2(M \otimes M \otimes M^{op})$ (and its symmetric $(1 \otimes (1 \otimes O)\delta_{(1)})^{*1}$, both defined as in section 2.1.2).

Lemma 27. *Consider $\delta_{(1)}$ satisfying $0_{\pi_1, \varrho_1}^a$ and $r, r' \in [2, \infty]$, $s, s' \in (1, \pi_1]$ with $1/s'' := \max(1/s + 1/r', 1/s' + 1/r, 1/r + 1/r' + 1/\varrho_1) \leq 1$. Consider $(1 \otimes \mathcal{O})U \in D(\delta_{(1)}^{r,s}) \otimes_{alg} D(\delta_{(1)}^{O_3;(1,1);2,s'})$.*

*Then $U \in D((\delta_{(1)} \otimes 1)^{*1})$ and $(\delta_{(1)i} \otimes 1)^{*1}(U) = U \#_1 \delta_{(1)i}^* 1 \otimes 1 - [m \circ 1 \otimes \tau \otimes 1] \otimes 1(\delta_{(1)}^{r,s} \otimes 1 \otimes 1 + 1 \otimes \delta_{(1)}^{O_3;(1,1);2,s'} \mathcal{O}) \in L^{s''}$.* \blacksquare

Proof of Proposition 24. The reader should maybe assume $C, D = 0$ (exact coassociativity) in the computations bellow in first reading.

We assume throughout $x \in B_{\Pi,R}^3(M_s, \delta, \delta_{(1)}, \delta_{(2)})$. Since $x \in B_{\pi(2),\rho'}^2(M_s, \delta_{(1)}, \delta_{(2)})$, lemma 16 with $r = \pi(2)$, $s = \rho'$ gives $x \in D(\delta_{(1)}^* \delta_{(2)}(x))$, with $\delta_{(1)}^* \delta_{(2)}(x) \in L^{\hat{\rho}}$ and :

$$\delta_{(1)j}^* \delta_{(2)j}(x) = \delta_{(2)}(x) \# \delta_{(1)j}^*(1 \otimes 1) - m(1 \otimes \tau \otimes 1)(\delta_{(1)j}^{\pi(2),\rho'} \otimes 1 + 1 \otimes \delta_{(1)j}^{\pi(2),\rho'})(\delta_{(2)j}(x)).$$

Now, from our assumption, we see we can apply lemma 15 to each term above to get :

$$\begin{aligned}
& \delta_i^{O;\tilde{\rho},\tilde{\sigma}}(\delta_{(1)j}^{1*}\delta_{(2)j}(x)) = ((1 \otimes O)\delta_{(2)j}(x))\# \delta_i^{O;\varrho^1,\varsigma^1}\delta_{(1)j}^{1*}(1 \otimes 1) + \\
& \delta_i^{O;\pi^{(2)},\rho} \otimes 1 \delta_{(2)j}(x) \#_2 \delta_{(1)j}^*(1 \otimes 1) + 1 \otimes \delta_i^{O;\pi^{(2)},\rho} \delta_{(2)j}(x) \#_1 \delta_{(1)j}^*(1 \otimes 1) \\
& - [m \circ 1 \otimes \tau \otimes 1] \otimes 1 \left(1 \otimes \delta_i^{O;(2,0);\rho',\sigma} \circ (\delta_{(1)j}^{\pi^{(2)},\rho'} \otimes 1 + 1 \otimes \delta_{(1)j}^{\pi^{(2)},\rho'}) (\delta_{(2)j}(x)) \right) \\
& - 1 \otimes [m \circ 1 \otimes \tau \otimes 1] \left(\delta_i^{O;(1,1);\rho',\sigma} \otimes 1 \circ (\delta_{(1)j}^{\pi^{(2)},\rho'} \otimes 1 + 1 \otimes \delta_{(1)j}^{\pi^{(2)},\rho'}) (\delta_{(2)j}(x)) \right)
\end{aligned}$$

Here for $U \in L^\rho(M \otimes M^{op}) \otimes_{alg} L^{\pi(2)}(M)$, $V \in L^{\varrho^2}(M)$ we wrote $U \#_1 V$ the map induced from the case $U = A \otimes b$ $U \#_1 V = (1 \otimes (O(Vb))).A$, and the symmetric case for $\#_2$ (m is used once for multiplication induced by the one in M , and once for the one induced from $a \in M^{op}$, $b \in M$ $m(a \otimes b) = O(b)a$).

Now we want to use coassociativity. First note that what is assumed on $\delta_{(2)j}(x)$ is exactly what is needed to apply lemma 25 (3) ($c = 1$, using also remark 21 to get the assumption). Thus, we have the two following identities :

$$\begin{aligned}
& 1 \otimes [m \circ 1 \otimes \tau \otimes 1] \left(\delta_i^{O;(1,1);\rho',\sigma} \otimes 1 \circ (\delta_{(1)j}^{\pi^{(2)},\rho'} \otimes 1) (\delta_{(2)j}(x)) \right) \\
& = 1 \otimes [m \circ 1 \otimes \tau \otimes 1] \left((\delta_{(1),j}^{O_2;(2,0);2,\sigma} \otimes 1) (\mathcal{O} \delta_i^{O;\pi^{(2)},\rho} \otimes 1 (\delta_{(2)j}(x))) \right) \\
& + 1 \otimes O[m \circ 1 \otimes \tau \otimes 1] \left(\sum_{k=1}^N ((\mathcal{O} \delta_k^{O;\pi^{(2)},\rho} \otimes 1) (\delta_{(2)j}(x)) \#_1 C_{i,j(1)}^k) \right), \\
& [m \circ 1 \otimes \tau \otimes 1] \otimes 1 \left(1 \otimes \delta_i^{O;(2,0);\rho',\sigma} \circ (1 \otimes \delta_{(1)j}^{\pi^{(2)},\rho'}) (\delta_{(2)j}(x)) \right) \\
& = [m \circ 1 \otimes \tau \otimes 1] \otimes 1 \left((1 \otimes \delta_{(1),j}^{O_3(1,1);2,\sigma} (1 \otimes \mathcal{O} \delta_i^{O;\pi^{(2)},\rho} (\delta_{(2)j}(x)))) \right) \\
& - [m \circ 1 \otimes \tau \otimes 1] \otimes O \left(\sum_{k=1}^N ((1 \otimes \mathcal{O} \delta_k^{O;\pi^{(2)},\rho}) (\delta_{(2)j}(x)) \#_2 C_{j(1),i}^k) \right).
\end{aligned}$$

Using also a much easier commutation of the form $(A \otimes 1 \otimes 1)(1 \otimes B) = (1 \otimes 1 \otimes B)(A \otimes 1)$ (and agreement of various closures on common domain) we finally got :

$$\begin{aligned}
& \delta_i^{O;\bar{\rho},\bar{\sigma}}(\delta_{(1)j}^{1*}\delta_{(2)j}(x)) = ((1 \otimes O)\delta_{(2)j}(x))\# \delta_i^{O;\varrho^1,\varsigma^1}\delta_{(1)j}^{1*}(1 \otimes 1) \\
& \delta_i^{O;\pi(2),\rho} \otimes 1\delta_{(2)j}(x)\#_2\delta_{(1)j}^{1*}(1 \otimes 1) + 1 \otimes \delta_i^{O;\pi(2),\rho}\delta_{(2)j}(x)\#_1\delta_{(1)j}^{1*}(1 \otimes 1) \\
& - [m \circ 1 \otimes \tau \otimes 1] \otimes 1 \left((\delta_{(1),j}^{\pi(2),\rho'} \otimes 1 \otimes 1 + 1 \otimes \delta_{(1),j}^{O_3;(1,1);2,\sigma} \mathcal{O})(1 \otimes \delta_i^{O;\pi(2),\rho}(\delta_{(2)j}(x))) \right) \\
& + \sum_{k=1}^N [m \circ 1 \otimes \tau \otimes 1] \otimes O \left(((1 \otimes \mathcal{O}\delta_k^{O;\pi(2),\rho})(\delta_{(2)j}(x))\#_2C_{j(1),i}^k) \right) \\
& - 1 \otimes [m \circ 1 \otimes \tau \otimes 1] \left((\delta_{(1),j}^{O_2;(2,0);2,\sigma} \mathcal{O} \otimes 1 + 1 \otimes 1 \otimes \delta_{(1),j}^{\pi(2),\rho'}) (\delta_i^{O;\pi(2),\rho} \otimes 1(\delta_{(2)j}(x))) \right) \\
& - \sum_{k=1}^N 1 \otimes O[m \circ 1 \otimes \tau \otimes 1] \left(((\mathcal{O}\delta_k^{O;\pi(2),\rho} \otimes 1)(\delta_{(2)j}(x))\#_1C_{i,j(1)}^k) \right) \\
& = (1 \otimes (1 \otimes O)\delta_{(1)j})^{1*}(\delta_i^{O;\pi(2),\rho} \otimes 1\delta_{(2)j}(x)) + (\delta_{(1)j} \otimes 1)^{1*}(1 \otimes \delta_i^{O;\pi(2),\rho}\delta_{(2)j}(x)) \\
& + ((1 \otimes O)\delta_{(2)j}(x))\# \delta_i^{O;\varrho^1,\varsigma^1}\delta_{(1)j}^{1*}(1 \otimes 1) + A
\end{aligned}$$

In the last equation we summarized by A lines involving sums over k before and we used lemma 27 with $r = \pi(2), r' = \rho \geq 2, s = \rho', s' = \sigma$, with of course $s'' \leq \tilde{\sigma}$.

We can thus take a scalar product with $y \otimes z, y, z \in D(\overline{\delta_{(1)}}) \cap M$ and use twice almost coassociativity in the form of lemma 26 :

$$\begin{aligned}
& \langle \delta_i^{O;\bar{\rho},\bar{\sigma}}(\delta_{(1)j}^{1*}\delta_{(2)j}(x)), y \otimes O(z) \rangle = \langle (1 \otimes \overline{\delta_{(2)j}}) \circ \delta_i(x) + \sum_{k=1}^N \delta_k(x)\#D_{i,j(2)}^k, y \otimes \overline{\delta_{(1)j}}(z) \rangle \\
& + \langle (\overline{\delta_{(2)j}} \otimes 1) \circ \delta_i(x) - \sum_{k=1}^N \delta_k(x)\#D_{j(2),i}^k, \overline{\delta_{(1)j}}(y) \otimes z \rangle + \langle \delta_{(2)j}(x)\# \delta_i^{O;\varrho^1,\varsigma^1}\delta_{(1)j}^{1*}(1 \otimes 1), y \otimes z \rangle \\
& + \sum_{k=1}^N \langle [m \circ 1 \otimes \tau \otimes 1] \otimes 1 \left(((\overline{\delta_{(2)j}} \otimes 1)(\delta_k(x) - \sum_{l=1}^N \delta_l(x)\#D_{j(2),k}^l)\#_2C_{j(1),i}^k) \right), y \otimes z \rangle \\
& - \sum_{k=1}^N \langle 1 \otimes [m \circ 1 \otimes \tau \otimes 1] \left(((1 \otimes \overline{\delta_{(2)j}}) \circ \delta_k(x) + \sum_{l=1}^N \delta_l(x)\#D_{k,j(2)}^l)\#_1C_{i,j(1)}^k \right), y \otimes z \rangle
\end{aligned}$$

□

2.2. Sufficient conditions for the main Assumption.

2.2.1. Statement of result and first reductions. Let us sum up right now the assumptions we will need and our result. The first assumption will be stated in the case $\delta = \tilde{\delta}$. The second will be partially more general (but we assume a strong core property for Δ).

Assumption 1 : (a) ∂ is almost coassociative with defect $C = (C_{i,j}^k, C_{j,i}^k)$ $C_i^k = C_{i,i}^k$. Furthermore we assume $C_i, C_{i,j}^k \in B_{vN}^1(M_0) \otimes_{alg} B_{vN}^1(M_0) \otimes_{alg} B_{vN}^1(M_0)$. : (a') ∂ satisfy assumption $0_{p,\infty}$. Moreover, we suppose that (b) for $\Pi_p = (p, \infty, p, p)$ and some R , $Comp_{\Pi_p}(R, \varrho, \varsigma, \infty)$ holds with $\tilde{\rho} \geq 2$, $\partial_j^* 1 \otimes 1 \in D(\partial^{e,\varsigma})$ and $\partial_i^{O;e,\varsigma} \partial_j^* 1 \otimes 1 \in M_0 \overline{\otimes} M_0^{op}$, (c) $\mathcal{B} := B_{\Pi_p, R}^3(M_0, \delta, \delta, \delta)$ is a core for $\tilde{\partial} : L^2(M_0) \rightarrow (L^2(M_0) \otimes L^2(M_0))^N$.

Assumption 2 : (a) We write $\partial = (\partial_1, \dots, \partial_N)$ (sometimes written $\partial_{(\infty)}$), $\tilde{\partial} = (\partial_{N+1}, \dots, \partial_{2N})$ and a sequence for $Q \in \mathbb{N} \cap [Q_0, \infty)$, $\partial_{(Q)} = (\partial_{(Q)1}, \dots, \partial_{(Q)N})$. We have $D(\partial) = D(\tilde{\partial}) = D(\partial_{(Q)}) = \mathcal{D}$ and ∂ (resp $\partial_{(Q)}$) is almost coassociative with respect to $\tilde{\partial}$ with defect $C = C^{(\infty)} = (C_{i,j}^k, C_{j,i}^k)_{j \leq N, i, k > N}$ (resp $C^{(Q)} = (C_{i,j}^{(Q)k}, C_{j,i}^{(Q)k})_{j \leq N, i, k > N} \in (M_0 \otimes_{alg} M_0 \otimes_{alg} M_0^{op})^{2N^2}$). Furthermore we assume $\|C_{i,j}^k - C_{i,j}^{(Q)k}\|_{M_0 \overline{\otimes} M_0 \overline{\otimes} M_0^{op}} \rightarrow 0$ and $1 \otimes 1 \otimes (\tau \otimes 1 \circ \partial_l) C_{i,j}^{(Q)k}, 1 \otimes (1 \otimes \tau \circ \partial_l) \otimes 1(C_{i,j}^{(Q)k}), C_{i,j}^{(Q)k} \in (M_0 \overline{\otimes} M_0^{op}) \otimes_{alg} M_0^{op}$ and converge in $(M_0 \overline{\otimes} M_0^{op}) \hat{\otimes} M_0^{op}$ ($Q \rightarrow \infty$) to $\mathcal{J}_{i,j,l}^{(1),k}, \mathcal{J}_{i,j,l}^{(2),k}, \mathcal{J}_{i,j}^{(3),k}$ ($1 \otimes \tau \circ \partial_l$) $\otimes 1 \otimes 1(C_{j,i}^{(Q)k}), 1 \otimes (\tau \otimes 1 \circ \partial_l) \otimes 1(C_{j,i}^{(Q)k}), C_{j,i}^{(Q)k} \in M_0 \otimes_{alg} (M_0 \overline{\otimes} M_0^{op})$ converge in $M_0 \hat{\otimes} (M_0 \overline{\otimes} M_0^{op})$ to $\mathcal{J}_{j,i,l}^{(1),k}, \mathcal{J}_{j,i,l}^{(2),k}, \mathcal{J}_{j,i}^{(3),k}$, all this stated for any $i, k > N, j, l \leq N$. We also assume $1 \otimes \tau \circ \partial_i$ $i \leq N$ bounded as a map $L^2(M_0) \rightarrow L^2(M_0)$.

Moreover, we suppose that (b) for $\Pi_{p,\pi}^q = (p, q, \pi, \pi)$, $\pi \geq p$ and some R , $Comp_{\Pi_{p,\pi}^q}(R, \varrho, \varsigma, \tilde{\omega})$ holds with $\tilde{\rho} \geq 2$, for $j \leq N$ $\partial_j^* 1 \otimes 1 \in D(\tilde{\partial}^{O;e,\varsigma})$, $\tilde{\partial}_i^{O;e,\varsigma} \partial_j^* 1 \otimes 1 \in M_0 \overline{\otimes} M_0^{op}$, and ∂ (resp. $\partial^{(Q)}$) satisfy assumption $0_{\pi,\infty}^a$ (resp 0_{π}^a) and $\tilde{\partial}$ $0_{p,q}^{\partial \tilde{\omega}}$ with moreover the derivation to which $\tilde{\partial}$ is reducible is also ∂ (with coefficients (k_j, i_j, K_j, I_i) satisfying $(1 \otimes O)K_j \in D(\delta^{(1,1);2,2\pi(2)/(\pi(2)-2)}) \cap D(\delta^{(2,0);2,2\pi(2)/(\pi(2)-2)}) \subset D(\tilde{\delta} \otimes 1 \oplus 1 \otimes \tilde{\delta})$ and $1 \otimes \tilde{\delta}_j(I_i^{-1}), (\tilde{\delta}_j \otimes 1(I_i^{-1})) \in M \overline{\otimes} M \overline{\otimes} M^{op}$. Likewise, $\partial^{(Q)}$ is equivalent to $\tilde{\partial}$ and the value K_j^Q such that $\partial_j^{(Q)}(x) = \tilde{\partial}_{k_j}(x) \# K_j^Q$ (note we ask for the same k_j as above) satisfies $\|K_j^Q - K_j\|_{M \overline{\otimes} M^{op}} \rightarrow_{Q \rightarrow \infty} 0, \|\delta^{a;2,2\pi/(\pi-2)}(1 \otimes O)(K_j^Q - K_j)\|_{2\pi/(\pi-2)} \rightarrow_{Q \rightarrow \infty} 0, a = (1, 1) \text{ or } (2, 0)$. (b') $\tilde{\partial}_j^* 1 \otimes 1 \in L^2(M_0)$

(c) We also assume $Alg_{p,r}(\rho_1), Alg_{r,p}(\rho_2), \rho_1, \rho_2 \geq 2$. Moreover $\mathcal{B} := D(\tilde{\delta} \otimes 1 \oplus 1 \otimes \tilde{\delta} \circ \tilde{\delta}) \cap \cap_{Q \in [Q_0, \infty)} B_{\Pi_{p,\pi}^q, R}^3(M_0, \tilde{\delta}, \delta, \delta^{(Q)})$ and for every $Q \in [Q_0, \infty)$, $B_{p,r,\rho_1,\rho_2}^{2,sym}(M_0, \delta^{(Q)}, \tilde{\delta})$ are cores for $\partial \oplus \tilde{\partial}$ and even (c') \mathcal{B} is a core for $\partial^* \partial$

Note with respect to assumption 2 (c'), \mathcal{B} will be seen in the domain of $\partial^* \partial$ in lemma 31 as a consequence of other points in assumption 2. Let us emphasize we will also write \mathcal{B} instead of $\mathcal{B} * \mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle$ since by lemma 18 it satisfies the same core properties for the extensions.

Theorem 28. Under assumption 1, $\tilde{\delta}$ and Δ satisfy the stability of filtration properties and assumption $\Gamma_1(\omega = 0, C)$ for some finite constant C in the context $\delta = \tilde{\delta}$ (see corollary 32). Under assumption 2, $\tilde{\delta}, \tilde{\delta}$ and Δ satisfy the stability of filtration properties and assumption $\Gamma_1(\omega = 0, C)$ for some finite constant C (see corollary 32 and lemma 29).

The proof will be divided into a bunch of lemmas and is the object of the whole end of this section. In case $\delta = \tilde{\delta}$, (d)', (g) and (h) are obvious (using also corollary 17 for (d')). We

thus first check they are easy under assumption 2 (using a supplementary assumption first for (h) we will deduce from our proof of (e), (f) i.e. in lemma 31 (v)).

Using lemma 19, it will only remains to check (e) and (f). We will then start by the proof of the stated boundedness in case of assumption 1 in the next subsection. As explained after lemma 24, we will finally solve domain issues in the last subsection.

Lemma 29. *Under Assumption 2, (d'), (g) of $\Gamma_1(\omega = 0, C)$ are satisfied for some C . (h) is also satisfied if we also assume for any $x \in D(\Delta = \delta^* \delta)$ $\delta(x) \in D(\tilde{\delta} \otimes 1 \oplus 1 \otimes \tilde{\delta})$ and $\|\tilde{\delta} \otimes 1 \oplus 1 \otimes \tilde{\delta} \circ \delta(x)\|_2 \leq c\|x\|_2 + c'\|\Delta(x)\|_2$. Moreover $1 \otimes \tau \circ \delta_k, k \leq N$ are also bounded.*

Proof. First note that using remark 23, δ is also almost coassociative with respect to $\tilde{\delta}$ with same defect C . Note that assumption g is automatically deduced from the existence of K_j . d' is also clear with $\tilde{\delta}_{lj}^{\otimes 1}$ the closure of $1 \otimes \tilde{\delta}_l$, and $\tilde{\delta}_{lj}^{\otimes 2}$ the closure of $\tilde{\delta}_l \otimes 1$ (use again corollary 17).

Note also that the boundedness $1 \otimes \tau \circ \delta_k$ for the extension δ_k of ∂_k is obvious since by freeness $1 \otimes \tau(\delta_k)(a_1 b_1 \dots a_n) = a_1 b_1 \dots 1 \otimes \tau(\partial_k(a_n))$ for $\tau(a_i) = \tau(b_i) = 0, a_i \in M_0, b_i \in W^*(\{S_t^{(i)}\}_{t,i})$.

Let us emphasize those facts also automatically implies h). First compute for $x \in \mathcal{B}$ using we can use almost coassociativity on this space (lemma 26 for $\delta^{(Q)}$ and then taking a limit using $\pi \geq 2$ and limits for $K_j^{(q)}$) :

$$\begin{aligned} \|\tilde{\delta} \otimes \delta(x)\|_2^2 &= \sum_{j,i=1}^N \|\overline{\tilde{\delta}_i \otimes 1} \delta_j(x)\|_2^2 + \|\overline{1 \otimes \tilde{\delta}_i} \delta_j(x)\|_2^2 \\ &= \sum_{j,i=1}^N \|\overline{1 \otimes \tilde{\delta}_j} \tilde{\delta}_i(x) + \sum_{k=1}^N \tilde{\delta}_k(x) \# C_{j+N,i}^{k+N}\|_2^2 + \|\overline{\tilde{\delta}_j \otimes 1} \tilde{\delta}_i(x) - \sum_{k=1}^N \tilde{\delta}_k(x) \# C_{i,j+N}^{k+N}\|_2^2 \\ &= \sum_{j,i=1}^N \|\overline{1 \otimes \tilde{\delta}_j} \tilde{\delta}_i(x)\|_2^2 + \|\overline{\tilde{\delta}_j \otimes 1} \tilde{\delta}_i(x)\|_2^2 + \sum_{j,i=1}^N \|\sum_{k=1}^N \tilde{\delta}_k(x) \# C_{j+N,i}^{k+N}\|_2^2 + \|\sum_{k=1}^N \tilde{\delta}_k(x) \# C_{i,j+N}^{k+N}\|_2^2 \\ &+ \sum_{j,i,k=1}^N 2\Re(\overline{1 \otimes \tilde{\delta}_j} \tilde{\delta}_i(x), \tilde{\delta}_k(x) \# C_{j+N,i}^{k+N}) - 2\Re(\overline{\tilde{\delta}_j \otimes 1} \tilde{\delta}_i(x), \tilde{\delta}_k(x) \# C_{i,j+N}^{k+N}). \end{aligned}$$

And the terms of the last line can be bounded under assumption 2, after rewriting for $U \in M \otimes_{alg} M, \sum V_c \otimes c \in M \otimes_{alg} D(\delta) \otimes_{alg} M$:

$$\begin{aligned} &|\langle \sum U \# (V_c \otimes c), 1 \otimes \delta_j \tilde{\delta}_i(x) \rangle| \\ &= |\langle \sum U \# (1 \otimes 1 \otimes c), 1 \otimes \delta_j(\tilde{\delta}_i(x) \# (V_c^*)) - \tilde{\delta}_i(x) \# ((1 \otimes \delta_j)(V_c^*)) \rangle| \\ &= |\langle \sum U \# (1 \otimes c), 1 \otimes (\tau \otimes 1 \delta_j)(\tilde{\delta}_i(x) \# (V_c^*)) - \tilde{\delta}_i(x) \# (1 \otimes (\tau \otimes 1) \delta_j(V_c^*)) \rangle| \\ &\leq \|U\|_2 \|\tilde{\delta}_k(x)\|_2 \times \\ &\times \left(\|(\tau \otimes 1 \delta_j)\| \|\sum V_c \otimes c\|_{(M_0 \overline{\otimes} M_0^{op}) \hat{\otimes} M_0} + \|\sum (1 \otimes (1 \otimes \tau \delta_j))(V_c) \otimes c\|_{(M_0 \overline{\otimes} M_0^{op}) \hat{\otimes} M_0} \right). \end{aligned}$$

In the third line we can take a limit to get $V_c \in M \overline{\otimes} M$ since $\tilde{\delta}_i(x) \in L^p(M \overline{\otimes} M)$ ($p > 2$) and $(\tau \otimes 1 \delta_j)$ bounded on L^2 , and then take any V_c such that we also have $1 \otimes (1 \otimes \tau) \delta_j(V_c) \in$

$M \overline{\otimes} M^{op}$, and get this for any $U \in L^2(M \otimes M)$. Thus we can apply the inequality of the last line in case $U = \tilde{\delta}_k(x)$ in our previous computation.

We thus deduced the bound for any $x \in B_{\Pi_{p,\pi},R}^3(M, \tilde{\delta}, \delta, \delta)$:

$$\begin{aligned} \|\tilde{\delta}^{\otimes} \delta(x)\|_2^2 &\leq \sum_{j,i=1}^N \|\overline{1 \otimes \delta_j} \tilde{\delta}_i(x)\|_2^2 + \|\overline{\delta_j \otimes 1} \tilde{\delta}_i(x)\|_2^2 \\ &+ \|\tilde{\delta}(x)\|_2^2 \sum_{j,i=1}^N \left(\left(\sum_{k=1}^N \|C_{j+N,i}^{k+N}\|_{M \overline{\otimes} M \overline{\otimes} M^{op}}^2 \right) + \left(\sum_{k=1}^N \|C_{i,j+N}^{k+N}\|_{M \overline{\otimes} M \overline{\otimes} M^{op}}^2 \right) \right) + 2 \|\tilde{\delta}(x)\|_2^2 \sum_{j=1}^N \\ &\left\{ \left(\sum_{i,k=1}^N \left(\|(\tau \otimes 1 \delta_j)\| \|\mathcal{J}_{j+N,i}^{(3),k+N}\|_{(M_0 \overline{\otimes} M_0^{op}) \hat{\otimes} M_0} + \|\mathcal{J}_{j+N,i,j}^{(2),k+N}\|_{(M_0 \overline{\otimes} M_0^{op}) \hat{\otimes} M_0} \right)^2 \right)^{1/2} \right. \\ &\left. + \left(\sum_{i,k=1}^N \left(\|(1 \otimes \tau \delta_j)\| \|\mathcal{J}_{i,j+N}^{(3),k+N}\|_{M_0 \hat{\otimes} (M_0 \overline{\otimes} M_0^{op})} + \|\mathcal{J}_{i,j+N,j}^{(2),k+N}\|_{M_0 \hat{\otimes} (M_0 \overline{\otimes} M_0^{op})} \right)^2 \right)^{1/2} \right\} \end{aligned}$$

Since $B_{\Pi_{p,\pi},R}^3(M, \tilde{\delta}, \delta, \delta)$ is a core for Δ , using the supplementary assumption, we get that for any $x \in D(\Delta)$ $\delta(x) \in D(\tilde{\delta}^{\otimes})$ and the inequality above. \square

2.2.2. Boundedness for $(1 \otimes \tau) \circ \delta_k$ under assumption 1. Our next lemma extends lemma 10 in [11] to the almost coassociative case.

Lemma 30. *Assume Assumption 1. Let $Z \in M \cap D(\bar{\delta})$, then the following inequality holds :*

$$\begin{aligned} \|(1 \otimes \tau)(\bar{\delta}_i(Z))\|_2 &\leq \|Z\|_2 \left[\left(2\|\delta_i^*(1 \otimes 1)\| + \left\| \sum_k \|1 \otimes \tau \otimes 1(C_i^k)\|_{M \hat{\otimes} B_{vN}^1(M)} \right\| \right) \right. \\ &\left. + \left(\|\delta_i^*(1 \otimes 1)\|^2 + \|\delta_i^{O;\varrho,\varsigma} \delta_i^*(1 \otimes 1)\|_{M \overline{\otimes} M^{op}} + \sum_k \|1 \otimes \tau \otimes 1(C_i^k)\|_{M \hat{\otimes} B_{vN}^1(M)} \right)^{1/2} \right] \end{aligned}$$

As a consequence, $(1 \otimes \tau) \circ \bar{\delta}_i$ extends as a bounded map $L^2(M, \tau) \rightarrow L^2(M, \tau)$. (We will write D the quantity in huge bracket above.)

Proof. The only non-trivial statement is the first one. The proof follows the one of lemma 10 in [11], with the coassociativity assumption weakened. Moreover, by the core property, we can assume first $Z \in B_{\Pi_{p,R}}^3(M_0, \delta, \delta, \delta)$. We first use the formula for δ_i^* , Corollary 4.3 in [53] recalled in lemma 12, and weakened coassociativity (given by lemma 26) in the third line. Below \cdot is the multiplication induced from $M \otimes M$ (since this is under a trace on the second tensor, M or M^{op} doesn't matter)

$$\begin{aligned}
\|(1 \otimes \tau)(\bar{\delta}_i(Z))\|_2^2 &= \langle (1 \otimes \tau)(\bar{\delta}_i(Z)) \otimes 1, \bar{\delta}_i(Z) \rangle \\
&= \langle (1 \otimes \tau)(\bar{\delta}_i(Z))\delta_i^*(1 \otimes 1), Z \rangle - \langle (1 \otimes \tau)\bar{\delta}_i(1 \otimes \tau)(\bar{\delta}_i(Z)), Z \rangle \\
&= \langle (1 \otimes \tau)(\bar{\delta}_i(Z)b)\delta_i^*(1 \otimes 1), Z \rangle - \sum_k \langle (1 \otimes \tau)(\bar{\delta}_k(Z).(1 \otimes \tau \otimes 1(C_i^k)\#(1 \otimes 1))), Z \rangle \\
&\quad - \langle (1 \otimes \tau \otimes \tau)(1 \otimes \bar{\delta}_i \circ \bar{\delta}_i(Z)), Z \rangle
\end{aligned}$$

We can use an adjointness relation on the first term, and on the third, and apply after that the definition of δ_i^* on this third term. In the second line, we apply the derivation property on the third term and second term and write $B_k = 1 \otimes \tau \otimes 1(C_i^k)$, $flip(a \otimes b) = b \otimes a$.

(12)

$$\begin{aligned}
&\|(1 \otimes \tau)(\bar{\delta}_i(Z))\|_2^2 \\
&= \langle (1 \otimes \tau)(\bar{\delta}_i(Z)), Z\delta_i^*(1 \otimes 1)^* \rangle - \sum_k \langle (1 \otimes \tau)(\bar{\delta}_k(Z).B_k), Z \rangle \\
&\quad - \langle (\bar{\delta}_i(Z), Z \otimes \delta_i^*(1 \otimes 1)) \rangle \\
&= \langle (1 \otimes \tau)(\bar{\delta}_i(Z)), Z\delta_i^*(1 \otimes 1)^* \rangle - \langle (\bar{\delta}_i(Z^*Z) - \bar{\delta}_i(Z^*)Z, 1 \otimes \delta_i^*(1 \otimes 1)) \rangle \\
&\quad - \sum_k \langle (m \circ (1 \otimes \tau \otimes 1)(\bar{\delta}_k \otimes 1(Z flip(B_k))), Z) + \sum_k \langle Zm \circ (((1 \otimes \tau)\bar{\delta}_k) \otimes 1)(flip(B_k))), Z \rangle
\end{aligned}$$

Let us note moreover that in the first equality above we can use proposition 6 in [11] recalled earlier at the end of lemma 12 and compute from now on with $Z \in M \cap D(\bar{\delta})$.

But for $b \in B_{vN}^1(M)$ we also know $\|(1 \otimes \tau)\bar{\delta}_i(Zb)\|_2 \leq \|b\|_M \|\bar{\delta}_i(Z)\|_2 + \|Z\|_2 \|\bar{\delta}_i(b)\|_{M \otimes M}$ so that $R(Z) := \sup_{i, \|b\|_{B_{vN}^1(M)} \leq 1} \|(1 \otimes \tau)\bar{\delta}_i(Zb)\|_2 \leq \|\bar{\delta}(Z)\|_2 + \|Z\|_2 < \infty$. But now, using that $B_{vN}^1(M)$ is a Banach algebra (this is obvious from $\|x\|_{B_{vN}^1(M)} := \|x\| + \|\delta(x)\|_{(M \otimes M)^N}$), we get $\|(1 \otimes b)B_k\|_{M \hat{\otimes} B_{vN}^1(M)} \leq \|1 \otimes \tau \otimes 1(C_i^k)\|_{M \hat{\otimes} B_{vN}^1(M)} \|b\|_{B_{vN}^1(M)}$.

Applying our equation (12) above to Zb , we get (using $\delta_i^*(1 \otimes \delta_i^*(1 \otimes 1)) \in M$ from our assumptions and lemma 12 again):

$$\begin{aligned}
\|(1 \otimes \tau)\bar{\delta}_i(Zb)\|_2^2 &\leq \|Zb\|_2^2 (\|\delta_i^*(1 \otimes \delta_i^*(1 \otimes 1))\| + \sum_k \|1 \otimes \tau \otimes 1(C_i^k)\|_{M \hat{\otimes} B_{vN}^1(M)}) \\
&\quad + R(Z) \|b\|_{B_{vN}^1(M)} \left(\|Zb\delta_i^*(1 \otimes 1)^*\|_2 + \|\delta_i^*(1 \otimes 1)(Zb)^*\|_2 + \|Zb\|_2 \sum_k \|1 \otimes \tau \otimes 1(C_i^k)\|_{M \hat{\otimes} B_{vN}^1(M)} \right).
\end{aligned}$$

so that we get the concluding equation :

$$\begin{aligned}
R(Z)^2 &\leq R(Z) \|Z\|_2 \left(2\|\delta_i^*(1 \otimes 1)\| + \sum_k \|1 \otimes \tau \otimes 1(C_i^k)\|_{M \hat{\otimes} B_{vN}^1(M)} \right) \\
&\quad + \|Z\|_2^2 \left(\|\delta_i^*(1 \otimes \delta_i^*(1 \otimes 1))\| + \sum_k \|1 \otimes \tau \otimes 1(C_i^k)\|_{M \hat{\otimes} B_{vN}^1(M)} \right)
\end{aligned}$$

□

2.2.3. *Almost commutation of $\tilde{\delta}$ and Δ on an extended domain.* We are now ready to solve our main domain issues (to get (e) and (f)) in the next :

Lemma 31. (i) *Assume assumption 2 (a),(b),(b'),(c) then for any $x \in \mathcal{B}$ we have $x \in D(\Delta_j)$ and $\delta_j^* \delta_j^{(Q)}(x) - \Delta_j(x) \rightarrow 0$ weakly in L^2 . Moreover For any $x \in \mathcal{B} \cap D(\Delta^{1/2} \Delta_j)$, then $\delta_i(x) \in D(\Delta_j \otimes 1 + 1 \otimes \Delta_j)$ and :*

$$\begin{aligned} \tilde{\delta}_i \Delta_j(x) &= (1 \otimes \Delta_j + \Delta_j \otimes 1) \tilde{\delta}_i(x) + \sum_{k=N+1}^{2N} \tilde{\delta}_k(x) \# ((1 \otimes \delta_j^*) C_{i+N,j}^k - (\delta_j^* \otimes 1) C_{j,i+N}^k) \\ &+ \sum_{k=N+1}^{2N} [m \circ 1 \otimes \tau \otimes 1] \otimes 1 \left(2(\bar{\delta}_j \otimes 1)(\tilde{\delta}_k(x)) \#_2 C_{j,i+N}^k + \sum_{l=N+1}^{2N} (\tilde{\delta}_l(x) \# C_{j,k}^l) \#_2 C_{j,i+N}^k \right) \\ &- \sum_{k=N+1}^{2N} 1 \otimes [m \circ 1 \otimes \tau \otimes 1] \left(2((1 \otimes \bar{\delta}_j) \circ \tilde{\delta}_k(x)) \#_1 C_{i+N,j}^k + \sum_{l=N+1}^{2N} (\delta_l(x) \# C_{k,j}^l) \#_1 C_{i+N,j}^k \right) \\ &+ \delta_j(x) \# \tilde{\delta}_i^{O;e,\varsigma} \delta_j^*(1 \otimes 1) \end{aligned}$$

(ii) *Assume assumption 1. If $x \in D(\bar{\delta})$ (resp. $x \in D(\Delta)$) then so is $1 \otimes \tau(\bar{\delta}_i(x))$.*

(iii) *Assume assumption 1 (in which case $\tilde{\delta} = \delta$) or assumption 2. Then $D(\Delta^{3/2}) \subset D(\bar{\Delta} \otimes 1 + 1 \otimes \bar{\Delta} \circ \tilde{\delta})$ and moreover we have for any $x \in D(\Delta^{3/2})$ the equation of (i) after summation over j .*

(iv) *Assume the assumptions of proposition 24, with $\tilde{\sigma}, \tilde{\rho} \geq 2$. Assume also there exists $\delta_{(3)}$ satisfying assumption 0' with $\delta_{(1)}^* \delta_{(2)} = \delta_{(3)}^* \delta_{(3)}$, and $\delta_{(3)}, \delta, \delta_{(1)}$ and $\delta_{(2)}$ equivalent. Also assume B^3 is a core for $\delta \delta_{(1)}^* \delta_{(2)}$ and $D_{j(2),i}^k \in D(\delta_{(1)j}^* \otimes 1), D_{i,j(2)}^k \in D(1 \otimes \delta_{(1)j}^*)$, $\delta_i^{O;e1,\varsigma1} \delta_{(1)}^* 1 \otimes 1, \delta_{(1)j}^* \otimes 1 D_{j(2),i}^k, 1 \otimes \delta_{(1)j}^* D_{i,j(2)}^k \in M \overline{\otimes} M^{op}$, then for any $x \in D(\delta \delta_{(1)}^* \delta_{(2)})$, $\delta_i(x) \in D(\delta_{(1)}^* \delta_{(2)} \otimes 1 + 1 \otimes \delta_{(1)}^* \delta_{(2)})$ and the relation in (i) holds mutatis mutandis. Moreover, for any $x \in D(\delta \delta_{(1)}^* \delta_{(2)})$,*

$$\|(\delta_{(3)} \otimes 1 \oplus 1 \otimes \delta_{(3)})(\delta(x))\|_2^2 \leq c_1 |\langle \delta \delta_{(3)}^* \delta_{(3)}(x), \delta(x) \rangle| + c_2 \|\delta_{(3)}(x)\|_2^2.$$

Finally, this last inequality for $\delta_{(i)} = \delta, \tilde{\delta}$ instead of δ is valid under the assumption of (iii).

(v) *Consider the assumptions of (iii) or (iv) (gathered with the notation of (iv)). Write $\tilde{\eta}_\alpha = \frac{\alpha}{\alpha + \delta_{(3)}^* \delta_{(3)}}, \tilde{\eta}_\alpha^\otimes = \frac{\alpha}{\alpha + 1 \otimes \delta_{(3)}^* \delta_{(3)} + \delta_{(3)}^* \delta_{(3)} \otimes 1}$. Then there exists a map $H_{i,\alpha}$ with $\|H_{i,\alpha}(x)\|_2 \leq c_4 \min(\alpha \|x\|_2, \sqrt{\alpha} \|\delta(x)\|_2)$ such that for any $Z \in D(\delta) \tilde{\eta}_\alpha^\otimes \delta_i(Z) - \delta_i \tilde{\eta}_\alpha(Z) = \frac{1}{\alpha} \tilde{\eta}_\alpha^\otimes H_{i,\alpha}(Z)$. Moreover, $\sup_{\alpha \geq 1} |\langle \delta_i \delta_{(3)}^* \delta_{(3)}(\tilde{\eta}_\alpha(Z)), \delta_i(\tilde{\eta}_\alpha(Z)) \rangle| \leq c_4 \|Z\|_2^2$. Finally, $D(\Delta) \subset D((\delta_{(3)} \otimes 1 \oplus 1 \otimes \delta_{(3)}) \delta)$, and for any $x \in D(\Delta)$, $\|(\delta_{(3)} \otimes 1 \oplus 1 \otimes \delta_{(3)})(\delta(x))\|_2 \leq c_4 (\|x\|_2 + \|\Delta(x)\|_2)$.*

Note that (i) and the beginning of (iv) are stated in order to be used later in section 4 to check (c') in assumption 2.

Proof. (i) First, note assumption 2 (a),(b),(b'),(c) is weaker than either assumption 1 or 2 (for 1, taking $\delta^{(Q)} = \delta$).

Let us start by applying proposition 24 to $U = V \# K_j^*$ for $V \in (D(\bar{\delta}) \cap M) \otimes_{alg} (D(\bar{\delta}) \cap M)$, $x \in \mathcal{B}$ (note that it implies $x \in D(\delta^* \delta^{(Q)})$ since $\tilde{\rho} \geq 2$). We apply it

with $i = k_j$ given by reducibility of $\tilde{\delta}$ to δ (K_j comes also from there, and recall $\tilde{\delta}_{k_j}(y) \# K_j = \delta_j(y)$, the proposition can be applied since we assume $(1 \otimes O)K_j \in D(\delta^{(1,1);2,2\pi(2)/(\pi(2)-2)}) \cap D(\delta^{(2,0);2,2\pi(2)/(\pi(2)-2)}) \subset D(\overline{\delta \otimes 1 \oplus 1 \otimes \delta})$, $K_j \in M \otimes M^{op}$ so that, using the last statement in lemma 12 applied to $\delta \otimes 1 \oplus 1 \otimes \delta$ as seen as a derivation on $M \otimes M^{op}$, with its range given the corresponding bimodule structure, we can extend the values of U in the proposition to this U). Thus we got :

$$\begin{aligned}
\langle (\delta_j^* \delta_j^{(Q)}(x)), \delta_j^*(V) \rangle &= \langle ((1 \otimes O)\delta_j^{(Q)}(x)) \# \tilde{\delta}_{k_j}^{Q;\varrho_1, \varsigma_1} \delta_j^*(1 \otimes 1), (1 \otimes O)(V \# K_j^*) \rangle \\
&+ \langle (1 \otimes \overline{\delta_j^{(Q)}}) \circ \tilde{\delta}_{k_j}(x) + \sum_{k=1}^N \tilde{\delta}_k(x) \# C_{k_j+N,j}^{(Q)k}, V \# (\delta^{(2,0);2,2\pi(2)/(\pi(2)-2)}(K_j^*) + (1 \otimes \overline{\delta_j})(V) \#_1 K_j^*) \rangle \\
&+ \langle (\overline{\delta_j^{(Q)}} \otimes 1) \circ \tilde{\delta}_{k_j}(x) - \sum_{k=1}^N \tilde{\delta}_k(x) \# C_{j,k_j}^{(Q)k}, V \# (\delta^{(1,1);2,2\pi(2)/(\pi(2)-2)}(K_j^*) + (\overline{\delta_j} \otimes 1)(V) \#_2 K_j^*) \rangle \\
&- \sum_{k=1}^N \langle 1 \otimes [m \circ 1 \otimes \tau \otimes 1] \left(((1 \otimes \overline{\delta_j^{(Q)}}) \circ \tilde{\delta}_k(x)) \#_1 C_{k_j,j}^k + \left(\sum_{l=1}^N \tilde{\delta}_l(x) \# C_{k,j}^{(Q)l} \right) \#_1 C_{k_j+N,j}^k \right), V \# K_j^* \rangle \\
&+ \sum_{k=1}^N \langle [m \circ 1 \otimes \tau \otimes 1] \otimes 1 \left(((\overline{\delta_j^{(Q)}} \otimes 1)(\tilde{\delta}_k(x)) \#_2 C_{j,k_j+N}^k - \left(\sum_{l=1}^N \tilde{\delta}_l(x) \# C_{j,k}^{(Q)l} \right) \#_2 C_{j,k_j+N}^k \right), V \# K_j^* \rangle
\end{aligned}$$

Since $\rho' \geq 2$ (as implied by $Comp^3$ and $\tilde{\rho} \geq 2$) we can extend this to $V \in D(\delta^{\pi(2),\rho'}) \otimes_{alg} D(\delta^{\pi(2),\rho'})$ and thus apply it to $V = \delta_j^{(Q)}(x)$. In the right hand side, the limit $Q \rightarrow \infty$ is possible so that $\|\delta_j^* \delta_j^{(Q)}(x)\|_2$ is uniformly bounded and thus we get our first statement (using a boundedness of other lines detailed in the next corollary).

In case of assumption 1, what we will say bellow is also valid for $U \in (M \cap D(\Delta)) \otimes_{alg} (M \cap D(\Delta))$, otherwise assume $U \in \mathcal{B} \otimes_{alg} \mathcal{B}$. Now we apply again proposition 24 but this time to U . Note that $U \in \tilde{\delta}^*$ by lemma 12 with $\tilde{\delta}^* U = (\tilde{\delta}^{O,\tilde{\rho},\tilde{\sigma}})^*(1 \otimes O)U$ (where the equality is by the very definition of $\tilde{\delta}^{O,\tilde{\rho},\tilde{\sigma}}$). Moreover, using the same lemma (the variant with (Q) and a limit first) and our assumptions on C for $j \leq N, i, k > N$, $\tilde{\delta}_k(x) \# C_{i,j}^k \in D(1 \otimes \delta^*)$ and $1 \otimes \delta^*(\tilde{\delta}_k(x) \# C_{i,j}^k) = \tilde{\delta}_k(x) \# ((1 \otimes \delta^*)(C_{i,j}^k)) - 1 \otimes [m \circ 1 \otimes \tau \otimes 1](1 \otimes \overline{\delta_j} \circ \tilde{\delta}_k(x) \#_1 C_{i,j}^k)$. Using also an analogous symmetric case, we finally get the limit version of our previous equation with $1 \otimes 1$ instead of K_j .

(13)

$$\begin{aligned}
\langle \Delta_j(x), \tilde{\delta}_i^* U \rangle &= \sum_{k=N+1}^{2N} \left(\langle \tilde{\delta}_k(x) \# ((1 \otimes \delta_j^*) C_{i+N,j}^k - (\delta_j^* \otimes 1) C_{j,i+N}^k), U \rangle \right. \\
&\quad + 2 \langle [m \circ 1 \otimes \tau \otimes 1] \otimes 1 ((\bar{\delta}_j \otimes 1) \tilde{\delta}_k(x) \#_2 C_{j,i+N}^k) - 1 \otimes [m \circ 1 \otimes \tau \otimes 1] ((1 \otimes \bar{\delta}_j) \tilde{\delta}_k(x) \#_1 C_{i+N,j}^k), U \rangle \\
&\quad + \langle \tilde{\delta}_i(x), (1 \otimes \Delta_j + \Delta_j \otimes 1) U \rangle + \langle (1 \otimes O)(\delta_j(x)) \# \tilde{\delta}_i^{O;\varrho,\varsigma} \delta_j^{1*}(1 \otimes 1), (1 \otimes O)(U) \rangle \\
&\quad - \sum_{k=N+1}^{2N} \langle [m \circ 1 \otimes \tau \otimes 1] \otimes 1 \left(\sum_{l=N+1}^{2N} \tilde{\delta}_l(x) \# C_{j,k}^l \#_2 C_{j,i+N}^k \right), U \rangle \\
&\quad - \sum_{k=N+1}^{2N} \langle 1 \otimes [m \circ 1 \otimes \tau \otimes 1] \left(\sum_{l=N+1}^{2N} \delta_l(x) \# C_{k,j}^l \#_1 C_{i+N,j}^k \right), U \rangle
\end{aligned}$$

Moreover, if we also assume $x \in D(\Delta^{1/2} \Delta_j)$, replacing the left hand side by $\langle \tilde{\delta}_i \Delta_j(x), U \rangle$, with any $U \in (M \cap D(\Delta)) \otimes_{alg} (M \cap D(\Delta))$, a core for $\Delta \otimes 1 + 1 \otimes \Delta$, we get the first statement.

- (ii) Here we are under assumption 1. Let $x \in D(\bar{\delta})$ and take $x_n \in \mathcal{B}$ converging to x in $D(\bar{\delta})$ since this is a core. We can compute (using lemma 26 again):

$$\bar{\delta}_j(1 \otimes \tau)(\bar{\delta}_i(x_n)) = \delta_{i=j}(1 \otimes 1 \otimes \tau)(\bar{\delta}_i(x_n) \# C_i) + (1 \otimes [(1 \otimes \tau)(\bar{\delta}_i)])(\bar{\delta}_j(x_n)).$$

By the boundedness result of lemma 30, the right hand side converges and gives the result since $\bar{\delta}$ is closed.

For the second statement, consider equation (13) with $U = V \otimes 1, V \in \mathcal{B}$, x_n above (with $x \in D(\delta)$) and $\langle \Delta_j(x_n), \tilde{\delta}_i^* U \rangle$ replaced by $\langle \delta_j(x_n), \delta_j^{\varrho,\varsigma} \delta_i^* V \otimes 1 \rangle$. Note that $\delta_j^{\varrho,\varsigma} \delta_i^* V \otimes 1 = \delta_j(V) \delta_i^* 1 \otimes 1 + V \delta_j^{\varrho,\varsigma} \delta_i^* 1 \otimes 1 - \delta_j 1 \otimes \tau \delta_i(V) \in L^2(M \otimes M)$. Note also that $\langle 1 \otimes \tau \delta_i(x_n), \Delta V \rangle = \langle \bar{\delta}(1 \otimes \tau \delta_i(x_n)), \bar{\delta}(V) \rangle$ converges to the analog with x by what we just proved. Since the two first and two last lines in (13) are bounded with respect to $\|\delta(x)\|_2$ (cf. corollary 32 for an explicit bound), we can get the equation at the limit $x_n \rightarrow x$.

We thus got :

$$\langle (1 \otimes \tau \delta_i) \Delta(x), V \rangle = \langle \bar{\delta}(1 \otimes \tau \delta_i(x)), \bar{\delta}(V) \rangle + (2),$$

where (2) comes from the sum over j of remaining terms in (13). Now we can extend this from $V \in \mathcal{B}$ to $V \in D(\delta)$ and thus we obtain our result by definition of Δ .

- (iii) First consider the case of assumption 2. If we apply to equation (13) the core property in (c), we get first the case $x \in D(\Delta)$, e.g. $x \in D(\Delta^{3/2})$. Then moving δ_i^* by duality, we can get $U \in (D(\Delta) \cap M) \otimes_{alg} (D(\Delta) \cap M)$ instead of $U \in \mathcal{B} \otimes_{alg} \mathcal{B}$ and then $U \in D(\Delta \otimes 1 + 1 \otimes \Delta)$. This concludes to the statement in this case.

Now consider the case of assumption 1. Consider again this time the variant of equation (13) with $U \in (M \cap D(\Delta)) \otimes_{alg} (M \cap D(\Delta))$, and $x \in B^3(M)$. Everything reduces to $U = a \otimes b$. Using lemma 12 we have $\delta_i^*(U) = a \delta_i^*(1 \otimes 1) b - (1 \otimes \tau) \delta_i(a) b - a(\tau \otimes 1) \delta_i(b)$. But now, $a, b \in M \cap D(\Delta), (1 \otimes \tau) \delta_i(a), (\tau \otimes 1) \delta_i(b) \in D(\Delta)$ by (ii) thus lemma 14 proves $-(1 \otimes \tau) \delta_i(a) b - a(\tau \otimes 1) \delta_i(b) \in D(\bar{\Delta}^1)$. Then, let us

write, for any $U \in (M \cap D(\Delta)) \otimes_{alg} (M \cap D(\Delta))$, $\delta_i^*(U) = U \# \delta_i^*(1 \otimes 1) - V$ with $V \in D(\overline{\Delta^1})$ we can now rewrite our equation (using $\bar{\delta}_i$ is a derivation on $M \cap D(\bar{\delta}_i)$) to see $U \# \delta_i^*(1 \otimes 1) \in D(\bar{\delta}_i)$:

$$\langle \delta_j(x), \delta_j(U \# \delta_i^*(1 \otimes 1)) \rangle = \langle x, \overline{\Delta_j^1}(V) \rangle + \langle \bar{\delta}_i(x), [1 \otimes \Delta_j + \Delta_j \otimes 1](U) \rangle + (2)$$

Now, once again, since \mathcal{B} is a core for $\bar{\delta}_i$, we get this for any $x \in D(\Delta^{3/2}) \cap M \subset D(\bar{\delta}_i)$. Using the remark before lemma 14, we can rewrite $\langle x, \overline{\Delta_j^1}(V) \rangle = \langle \Delta_j(x), V \rangle$, and thus (only coming back to our original notations):

$$\langle \bar{\delta}_i \Delta_j(x), U \rangle = \langle \bar{\delta}_i(x), [1 \otimes \Delta_j + \Delta_j \otimes 1](U) \rangle + (2)$$

Finally, (using stability by $\phi_t \otimes \phi_t$) it is easily seen that $(M \cap D(\Delta)) \otimes_{alg} (M \cap D(\Delta))$ is a core for $\overline{\Delta \otimes 1 + 1 \otimes \Delta}$, and thus we can take U in the domain of that operator (since the right scalar product factor in (2) is bounded in $U \in L^2(M \otimes M)$ as in (ii)), and finally, since this operator is closed, we get our result.

- (iv) Applying proposition 24, with a slight extension in U as in (i), we can get it for $U = \delta(x)$ (note under our assumptions, especially $\tilde{\sigma} \geq 2$, $B^3 \subset D(\delta \delta_{(3)}^* \delta_{(3)})$). From the resulting equation, we thus readily obtains the inequality (using the core property to extend it). Considering back the case of a general U in proposition 24, our inequality and the core property imply the fact that we can take the limit to get any $x \in D(\delta \delta_{(3)}^* \delta_{(3)})$ and this concludes as in (i). In the case of the assumption of (iii), we can start directly from the equation in (i) and take a scalar product with $\delta(x)$ to get the inequality as before.
- (v) We write $\tilde{\Delta} = \delta_{(3)}^* \delta_{(3)}$. Since for any $Z \in D(\delta)$, $\tilde{\eta}_\alpha(Z) \in D(\tilde{\Delta}^{3/2})$, we can apply (iii) or (iv) to get the equation in (i) in the form $\delta_i \tilde{\Delta} \tilde{\eta}_\alpha(Z) = (\tilde{\Delta} \otimes 1 + 1 \otimes \tilde{\Delta}) \delta_i \tilde{\eta}_\alpha(Z) + H_i(\tilde{\eta}_\alpha(Z))$ with $\|H_i(\tilde{\eta}_\alpha(Z))\|_2^2 \leq c'_1 |\langle \delta \tilde{\Delta}(\tilde{\eta}_\alpha(Z)), \delta(\tilde{\eta}_\alpha(Z)) \rangle| + c'_2 \|\delta_{(3)}(\tilde{\eta}_\alpha(Z))\|_2^2$. Let us call C the constant given by equivalence $\|\delta(Z)\|_2 \leq C \|\delta_{(3)}(Z)\|_2$. We obviously take $H_{i,\alpha} = H_i \tilde{\eta}_\alpha(Z)$, thus we can first note $\|H_{i,\alpha}(Z)\|_2^2 \leq (2\alpha c'_1 C^2 + c'_2) \|\delta_{(3)}(Z)\|_2^2 \leq \alpha C'^2 \|\delta_{(3)}(Z)\|_2^2$ implying one of the bounds for $H_{i,\alpha}$.

We get in the spirit of lemma 2 for any $Z \in D(\delta)$: $\tilde{\eta}_\alpha^\otimes \delta_i(Z) - \delta_i \tilde{\eta}_\alpha(Z) = \frac{1}{\alpha} \tilde{\eta}_\alpha^\otimes H_i \tilde{\eta}_\alpha(Z)$.

For $Z \in D(\delta)$, we call

$R(Z) = \sup_{\alpha \geq 1} |\langle \delta \tilde{\Delta}(\tilde{\eta}_\alpha(Z)), \delta(\tilde{\eta}_\alpha(Z)) \rangle| / \alpha^2 \leq C'' \|\tilde{\Delta}^{1/2}(Z)\|_2 \|Z\|_2 < \infty$ and then $S(Z) = \sup_n \sup_{Y=T_{\alpha_1} \dots T_{\alpha_n}(Z), \alpha_i \geq 1} R(Y)$ where we define $T_\alpha = \tilde{\Delta} \tilde{\eta}_\alpha / 2\alpha$ the contraction commuting with $\tilde{\Delta}^{1/2}$ so that we also get $S(Z) \leq C'' \|\tilde{\Delta}^{1/2}(Z)\|_2 \|Z\|_2 < \infty$. Using the formula $\eta_\alpha^{1/2} = \pi^{-1} \int_0^\infty \frac{t^{-1/2}}{1+t} \eta_{\alpha(1+t)/t} dt$ (cf. lemma 2.2 in [36], for brevity we write $\alpha_t = \alpha(1+t)/t \geq \alpha$), one deduces

$$\delta_i \tilde{\eta}_\alpha^{1/2}(Z) = \tilde{\eta}_\alpha^{\otimes 1/2} \delta_i(Z) - \frac{1}{\alpha \pi} \int_0^\infty \frac{t^{1/2}}{(1+t)^2} \tilde{\eta}_{\alpha_t}^\otimes H_i \tilde{\eta}_{\alpha_t}(Z) dt.$$

We can compute (for $\alpha \geq 1$)

$$\begin{aligned}
|\langle \delta_i \tilde{\Delta}(\tilde{\eta}_\alpha(Z)), \delta_i(\tilde{\eta}_\alpha(Z)) \rangle| &\leq |\langle \delta_i \tilde{\eta}_\alpha^{1/2} \tilde{\Delta}(\tilde{\eta}_\alpha(Z)), \delta_i(\tilde{\eta}_\alpha^{1/2}(Z)) \rangle| + \frac{1}{\alpha\pi} \int_0^\infty dt \frac{t^{1/2}}{(1+t)^2} \times \\
&\times |\langle \tilde{\eta}_{\alpha_t}^\otimes H_i \tilde{\eta}_{\alpha_t} \tilde{\Delta} \tilde{\eta}_\alpha(Z), \delta_i(\tilde{\eta}_\alpha^{1/2}(Z)) \rangle - \langle \tilde{\eta}_{\alpha_t}^\otimes \delta_i \tilde{\Delta} \tilde{\eta}_\alpha(Z), H_i \tilde{\eta}_{\alpha_t}(\tilde{\eta}_\alpha^{1/2}(Z)) \rangle| \\
&\leq C^2(2\alpha)^2 \|Z\|_2^2 + \int_0^\infty dt \frac{t^{1/2} C}{\alpha\pi(1+t)^2} 2\alpha^{1/2} (\sqrt{c_2} 2\alpha(\alpha_t)^{1/2} \|Z\|_2^2 + \sqrt{c_1} \sqrt{R(T_\alpha(Z))} \|Z\|_2 2\alpha\alpha_t) \\
&+ \frac{1}{\alpha\pi} \int_0^\infty dt \frac{t^{1/2}}{(1+t)^2} |\langle (\delta_i \tilde{\eta}_{\alpha_t} + \frac{1}{\alpha_t} \tilde{\eta}_{\alpha_t}^\otimes H_i \tilde{\eta}_{\alpha_t}) \tilde{\Delta} \tilde{\eta}_\alpha(Z), H_i \tilde{\eta}_{\alpha_t}(\tilde{\eta}_\alpha^{1/2}(Z)) \rangle| \\
&\leq C^2(2\alpha)^2 \|Z\|_2^2 + D\alpha^{3/2} \sqrt{R(T_\alpha(Z))} \|Z\|_2 + E\alpha^{3/2} \|Z\|_2^2 \\
S(Z) &\leq D' \|Z\|_2^2 + E' \sqrt{S(Z)} \|Z\|_2.
\end{aligned}$$

We thus deduce, $H_{i,\alpha}$ bounded by $c_4\alpha$. As a consequence for any x , $\|(\delta_{(3)} \otimes 1 \oplus 1 \otimes \delta_{(3)})(\delta(\eta_1(x)))\|_2$ is finite and taking for $x \in D(\Delta)$ this equation for $\Delta(x)$, and using also $\eta_1\Delta(x) = x - \eta_1(x)$ we get the last statement. \square

Putting everything together, we can now conclude (under assumption 1 or 2) :

Corollary 32. *There exists a bounded operator \mathcal{H} on $L^2(M \otimes M)^{\oplus N}$ keeping invariant, for any s , $L^2(M_s \otimes M_s)^{\oplus N}$ such that for any $x \in D(\tilde{\delta} \circ \Delta)$:*

$$\Delta^\otimes \circ \tilde{\delta}(x) - \tilde{\delta} \circ \Delta(x) = \mathcal{H}(\tilde{\delta}(x)).$$

Explicitly, seeing \mathcal{H} as $(\mathcal{H}_{ij}) \in M_N(B(L^2(M \otimes M)))$ we have (the extension by continuity beyond $D(\tilde{\delta}) \otimes_{alg} D(\tilde{\delta})$ following from lemma 30 or assumption 2, $C_{ii} := C_i$, we use the notation of assumption 2, in case of assumption 1, it suffices to define $C_{i+N,j}^{k+N} = C_{i,j}^k$, $C_{j,i+N}^{k+N} = C_{j,i}^k$):

$$\begin{aligned}
\mathcal{H}_{ij}(a \otimes b) &= \sum_{k=1}^N a(\delta_k^* \otimes 1)(C_{k,i+N}^{j+N})b - a(1 \otimes \delta_k^*)(C_{i+N,k}^{j+N})b - a(\bar{\delta}_i \delta_j^* 1 \otimes 1)b \\
&- \sum_{k=1}^N 2(m \circ ((1 \otimes \tau \delta_k) \otimes 1) \otimes 1)(a C_{k,i+N}^{j+N} b) \\
&+ \sum_{k=1}^N 2(a(m \circ ((1 \otimes \tau \delta_k) \otimes 1) \otimes 1)(C_{k,i+N}^{j+N}))b + \sum_{l=1}^N a(1 \otimes \tau \otimes 1)(C_{l,k+N}^{j+N} \# (\sigma \otimes 1)(C_{l,i+N}^{k+N}))b \\
&+ \sum_{k=1}^N 2(1 \otimes m \circ (1 \otimes (\tau \otimes 1 \delta_k)))(a C_{i+N,k}^{j+N} b) + \sum_{l=1}^N a(1 \otimes \tau \otimes 1)(C_{k+N,l}^{j+N} \# (1 \otimes \sigma)(C_{i+N,l}^{k+N}))b \\
&- \sum_{k=1}^N 2(a(1 \otimes m \circ (1 \otimes (\tau \otimes 1 \delta_k)))(C_{i+N,k}^{j+N})b).
\end{aligned}$$

Here we wrote $a \otimes b \otimes c \# a' \otimes b' \otimes c' = aa' \otimes bb' \otimes c'c$. As a consequence, we get, with $\|1 \otimes \tau \otimes \delta_k\|$ bounded either by lemma 30 or assumption 2:

$$\begin{aligned} \|\mathcal{H}\|^2 &\leq \sum_{ij} \|\mathcal{H}_{ij}\|^2 \leq \sum_{ij} \left(\left\| \sum_k (\delta_k^* \otimes 1)(C_{k,i+N}^{j+N}) - (1 \otimes \delta_k^*)(C_{i+N,k}^{j+N}) - (\bar{\delta}_i \delta_j^* 1 \otimes 1) \right\|_{M \bar{\otimes} M^{op}} \right. \\ &\quad + \left\| \sum_k 2((m \circ ((1 \otimes \tau \delta_k) \otimes 1) \otimes 1)(C_{k,i+N}^{j+N})) + \sum_l (1 \otimes \tau \otimes 1)(C_{l,k+N}^{j+N} \# (\sigma \otimes 1)(C_{l,i+N}^{k+N})) \right\|_{M \bar{\otimes} M^{op}} \\ &\quad + \left\| \sum_k 2((1 \otimes m \circ (1 \otimes (\tau \otimes 1 \delta_k)))(C_{i+N,k}^{j+N})) - \sum_l (1 \otimes \tau \otimes 1)(C_{k+N,l}^{j+N} \# (1 \otimes \sigma)(C_{i+N,l}^{k+N})) \right\|_{M \bar{\otimes} M^{op}} \\ &\quad \left. + 2 \sum_k \|1 \otimes (1 \otimes \tau \delta_k)\| \left(\|C_{k,i+N}^{j+N}\|_{M \hat{\otimes} (M \bar{\otimes} M^{op})} + \|C_{i+N,k}^{j+N}\|_{(M \bar{\otimes} M^{op}) \hat{\otimes} M} \right)^2 \right). \end{aligned}$$

■

3. COMPLEMENTARY PROPERTIES OF OUR MAIN EXAMPLE

3.1. A Ito Formula for resolvents under weak assumptions. Let us consider an integral of the form :

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t U_s \# dS_s$$

where $X_0 \in M_0 = W$, $s \mapsto K_s$ weakly measurable with $K_s \in L^1(M_s)$, $\int_0^T \|K_s\|_1 ds < \infty \forall T > 0$ and $U \in \mathcal{B}_2^a$. We also assume $K_s = K_s^*$, $U_s = U_s^*$ (in this part we use the involution induced by $HS(M)$, i.e. $(a \otimes b)^* = b^* \otimes a^*$), $X_0 = X_0^*$ so that $X_t = X_t^*$.

We would like to find a formula for $(z + X_t)^{-1}$, $z \in \mathbb{C}$, $\Im z > 0$, to compute the Cauchy-transform of X_t with this unbounded $X_t \in L^1(M_t)$. If we supposed $K_s \in M_s$, $U_s \in \mathcal{B}_\infty^a$ Proposition 4.3.4 of [2] would conclude (see this article for notations, the case with N free Brownian motions as in our case is similar to their case, especially we write in this part also $\#$ for multiplication in $M \otimes M^{op} \otimes M$ without confusion with the previous notation for multiplication in $M \otimes M^{op}$) since $f(x) = \frac{1}{z+x} = \int_{\mathbb{R}} e^{ixy} \mu(dy)$ with $\mu(dy) = -i1_{[0,\infty)} e^{izy} dy$ (which satisfy $\mathcal{I}_2(f) < \infty$, and thus their results apply).

But we are not in such a bad position because all the terms of their expression in the Ito Formula for $(z + X_t)^{-1}$ make sense, this almost only require applying a standard density argument left to the reader.

Proposition 33. *With the previous assumptions we have :*

$$\begin{aligned} (z + X_t)^{-1} &= (z + X_0)^{-1} - \int_0^t [(z + X_s)^{-1} \otimes (z + X_s)^{-1}] \# U_s \# dS_s \\ &\quad - \int_0^t [(z + X_s)^{-1} \otimes (z + X_s)^{-1}] \# K_s ds \\ &\quad + \sum_{i=1}^N \int_0^t m \circ 1 \otimes \tau \otimes 1 (1 \otimes U_s^{(i)} \# (z + X_s)^{-1} \otimes (z + X_s)^{-1} \otimes (z + X_s)^{-1} \# U_s^{(i)} \otimes 1) ds. \end{aligned}$$

The two next lemmas are also left to the reader.

Lemma 34. *Let*

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t U_s \# dS_s$$

where $X_0 \in M_0$, $s \mapsto K_s$ weakly measurable with $K_s \in L^1([0, T], L^1(M_s))$, and $U \in \mathcal{B}_2^a$. We also assume $X_t \in M$ (boundedly in t). Let say $\|X_t\| < 1$.

Then there exists $X_t^n = X_0 + \int_0^t K_s^n ds + \int_0^t U_s^n \# dS_s$ with $s \mapsto K_s^n$ weakly measurable with $K_s^n \in L^\infty([0, T]) \otimes M_s$, K^n converging to K in $L^1([0, T], L^1(M_s))$, and $U^n \in \mathcal{B}_\infty^a$, U^n converging to U in \mathcal{B}_2^a . Moreover we have $\|X_t^n\| \leq 1$.

The following variant of the Ito product formula (proposition 4.3.2 in [2]) is now obvious :

Lemma 35. *Let*

$$\begin{aligned} X_t &= X_0 + \int_0^t K_s ds + \int_0^t U_s \# dS_s \\ Y_t &= Y_0 + \int_0^t L_s ds + \int_0^t V_s \# dS_s \end{aligned}$$

where $X_0, Y_0 \in M_0$, $s \mapsto K_s, s \mapsto L_s$ weakly measurable with $K_s, L_s \in L^1([0, T], L^1(M_s))$, and $U, V \in \mathcal{B}_2^a$. We also assume $X_t, Y_t \in M$ (boundedly in t).

Then, for any $t \leq T$:

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \int_0^t X_s L_s + K_s Y_s ds + \int_0^t m \circ 1 \otimes \tau \circ m \otimes 1 (U_s \otimes V_s) ds \\ &\quad + \int_0^t X_s V_s + U_s Y_s \# dS_s \end{aligned}$$

■

3.2. Boundedness. In this part, we are interested in the example of part 2. Under assumption 0, we write $X_t \in \mathcal{B}_{2, \phi \bar{\delta}}^a$, $X_t^\epsilon \in \mathcal{B}_{2, \Delta^{1/2}}^a$ the solutions given by theorem 10 (i) and lemma 19 . We moreover consider an initial condition $X_0 \in M_0 \cap D(\bar{\delta})$.

Proposition 36. *With those assumptions, for any complex number z with $\Im z > 0$, $\frac{1}{z + X_s^\epsilon}$ is in $\mathcal{B}_{2, \Delta^{1/2}}^a$ and*

$$\begin{aligned} (z + X_t^\epsilon)^{-1} &= \phi_t((z + X_0^\epsilon)^{-1}) + (1 - \epsilon) \int_0^t \phi_{t-s}(\bar{\delta}((z + X_s^\epsilon)^{-1}) \# dS_s) \\ &\quad + ((1 - \epsilon)^2 - 1) \sum_{i=1}^N \int_0^t \phi_{t-s} (m \circ 1 \otimes \tau \otimes 1((z + X_s^\epsilon)^{-1} \bar{\delta}_i(X_s^\epsilon)(z + X_s^\epsilon)^{-1} \bar{\delta}_i(X_s^\epsilon)(z + X_s^\epsilon)^{-1})) ds. \end{aligned}$$

As a consequence, if we assume moreover $\|X_t\|_2 = \|X_0\|_2$ (a.e. t , this is the case e.g. for a mild solution given by Theorem 10 (ii)) then $X_t \in M$ for all t (recall we supposed $X_0 \in M$) and we also have $\|X_t\| \leq \|X_0\|$ (actually equal a.e.), and likewise for any $\epsilon > 0$, $X_t^\epsilon \in M$.

Proof. Let $\epsilon > 0$. Since we have a mild solution at ϵ level, by theorem 10 and since a mild solution is a weak solution as seen in Proposition 8, we get by a self-adjointness argument of Δ for characterizing its domain that $\int_0^t X_s^\epsilon ds \in D(\Delta)$ and :

$$X_t^\epsilon = X_0 - \frac{1}{2}\Delta \int_0^t X_s^\epsilon ds + (1 - \epsilon) \int_0^t \delta(X_s^\epsilon) \# dS_s.$$

Thus, taking a resolvent, using lemma 3 (ii), we deduce (any $\alpha > 0$) :

$$\eta_\alpha(X_t^\epsilon) = \eta_\alpha(X_0) - \frac{1}{2} \int_0^t \Delta \eta_\alpha(X_s^\epsilon) ds + (1 - \epsilon) \int_0^t \eta_\alpha^\otimes \delta(X_s^\epsilon) \# dS_s,$$

where $X_0 \in M_0$, $s \mapsto K_s = -\frac{1}{2}\Delta \eta_\alpha(X_s^\epsilon)$ weakly measurable with $K_s \in L^2(M_s)$, $\int_0^T \|K_s\|_2^2 ds < \infty \forall T > 0$ (all this using the definition of $\mathcal{B}_{2,\Delta^{1/2}}^a$ and $U_s = \eta_\alpha^\otimes \delta(X_s^\epsilon) \in \mathcal{B}_2^a$. Recall (again what we recalled at the beginning of section 2) that $\eta_\alpha(X_0) \in M$ by the general Dirichlet form theory implying η_α is a completely positive contraction on M . We are in position to apply proposition 33, thus we have :

$$\begin{aligned} (z + \eta_\alpha(X_t^\epsilon))^{-1} &= (z + \eta_\alpha(X_0^\epsilon))^{-1} \\ &- (1 - \epsilon) \int_0^t [(z + \eta_\alpha(X_s^\epsilon))^{-1} \otimes (z + \eta_\alpha(X_s^\epsilon))^{-1}] \# \eta_\alpha^\otimes \bar{\delta}(X_s^\epsilon) \# dS_s \\ &+ \frac{1}{2} \int_0^t [(z + \eta_\alpha(X_s^\epsilon))^{-1} \otimes (z + \eta_\alpha(X_s^\epsilon))^{-1}] \# \Delta \eta_\alpha(X_s^\epsilon) ds \\ &+ (1 - \epsilon)^2 \sum_{i=1}^N \int_0^t ds \\ &m \circ 1 \otimes \tau \otimes 1 ((z + \eta_\alpha(X_s^\epsilon))^{-1} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\epsilon) (z + \eta_\alpha(X_s^\epsilon))^{-1} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\epsilon) (z + \eta_\alpha(X_s^\epsilon))^{-1}). \end{aligned}$$

But remark that for any $x \in D(\bar{\delta})$, $(z+x)^{-1} \in D(\bar{\delta})$, and $\bar{\delta}((z+x)^{-1}) = -(z+x)^{-1} \bar{\delta}(x)(z+x)^{-1}$. Indeed, we check this easily on $D(\delta) \subset M$ by Leibniz rule, and taking $x_n \in D(\delta)$ converging to x in $D(\bar{\delta})$, a usual formula on resolvents $(z+x_n)^{-1} - (z+x)^{-1} = (z+x_n)^{-1}(x-x_n)(z+x)^{-1}$ get convergence of $(z+x_n)^{-1}$ to $(z+x)^{-1}$ in $L^2(M)$, and thus of $\bar{\delta}((z+x_n)^{-1})$ in $L^1(M \otimes M)$ to $(z+x)^{-1} \bar{\delta}(x)(z+x)^{-1}$. A fortiori, we have weak convergence in $L^2(M \otimes M)$. Since a convex set in $L^2(M) \oplus L^2(M \otimes M)$ is closed if and only if it is weakly closed by Hahn-Banach theorem, we get $(z+x)^{-1} \in D(\bar{\delta})$ and the result.

Analogously, we have for any $x \in D(\Delta)$, $(z+x)^{-1} \in D(\Delta^1)$ (cf. the paragraph before lemma 13 for definition) and moreover :

$$\begin{aligned} -\Delta^1((z+x)^{-1}) &= (z+x)^{-1} \Delta(x)(z+x)^{-1} \\ &+ 2 \sum_{i=1}^N m \circ 1 \otimes \tau \otimes 1 (1 \otimes \bar{\delta}_i(x) \# (z+x)^{-1} \otimes (z+x)^{-1} \otimes (z+x)^{-1} \# \bar{\delta}_i(x) \otimes 1). \end{aligned}$$

Let us write $R_{t,z,\alpha,\epsilon} = (z + \eta_\alpha(X_t^\epsilon))^{-1}$. Thus, we have obtained, if we apply this formula to our previous equation making appear terms by emphasizing "commutators" of η_α^\otimes and δ ,

writing $Y_{s,z,\alpha,\epsilon,i} := (R_{s,z,\alpha,\epsilon}(\eta_\alpha^\otimes \bar{\delta}_i(X_s^\epsilon) - \bar{\delta}_i \eta_\alpha(X_s^\epsilon))R_{s,z,\alpha,\epsilon} :$

$$\begin{aligned} R_{t,z,\alpha,\epsilon} &= R_{0,z,\alpha,\epsilon} + (1-\epsilon) \int_0^t (\bar{\delta}(R_{s,z,\alpha,\epsilon}) + Y_{s,z,\alpha,\epsilon}) \# dS_s - \frac{1}{2} \int_0^t \Delta^1(z + \eta_\alpha(X_s^\epsilon))^{-1} ds \\ &+ \sum_{i=1}^N \int_0^t m \circ 1 \otimes \tau \otimes 1(Y_{s,z,\alpha,\epsilon,i} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\epsilon) R_{s,z,\alpha,\epsilon} + R_{s,z,\alpha,\epsilon} \bar{\delta}_i \eta_\alpha(X_s^\epsilon) Y_{s,z,\alpha,\epsilon,i}) ds \\ &+ ((1-\epsilon)^2 - 1) \sum_{i=1}^N \int_0^t m \circ 1 \otimes \tau \otimes 1(R_{s,z,\alpha,\epsilon} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\epsilon) R_{s,z,\alpha,\epsilon} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\epsilon) R_{s,z,\alpha,\epsilon}) ds. \end{aligned}$$

As in the proof of proposition 7 (take $\zeta \in D(\Delta) \cap M$) showing that a strong solution is a mild solution, we have :

$$\begin{aligned} R_{t,z,\alpha,\epsilon} &= \phi_t(R_{0,z,\alpha,\epsilon}) + (1-\epsilon) \int_0^t \phi_{t-s}^\otimes (\bar{\delta}(R_{s,z,\alpha,\epsilon}) + Y_{s,z,\alpha,\epsilon}) \# dS_s \\ &+ \sum_{i=1}^N \int_0^t \phi_{t-s} (m \circ 1 \otimes \tau \otimes 1(Y_{s,z,\alpha,\epsilon,i} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\epsilon) R_{s,z,\alpha,\epsilon} + R_{s,z,\alpha,\epsilon} \bar{\delta}_i \eta_\alpha(X_s^\epsilon) Y_{s,z,\alpha,\epsilon,i}) ds \\ &+ ((1-\epsilon)^2 - 1) \sum_{i=1}^N \int_0^t \phi_{t-s} m \circ 1 \otimes \tau \otimes 1(R_{s,z,\alpha,\epsilon} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\epsilon) R_{s,z,\alpha,\epsilon} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\epsilon) R_{s,z,\alpha,\epsilon}) ds. \end{aligned}$$

We now want to make α tend to ∞ . The three terms with Y tend to zero by dominated convergence theorem (domination modulo constant by $\|\bar{\delta}(X_s^\epsilon)\|_2^2$ since $X^\epsilon \in \mathcal{B}_{2,\Delta^{1/2}}^a$.) In the last line we can remove η_α^\otimes in the same way and we get weak convergence in L^1 to the expected limit like in the end of the proof of proposition 33 (of course we have to use ϕ bounded on M). Clearly, the two resolvents in the first line converge in L^2 and the same kind of reasoning already made shows that $\bar{\delta}((z + \eta_\alpha(X_s^\epsilon))^{-1})$ weakly converge in L^2 to $\bar{\delta}((z + X_s^\epsilon)^{-1})^3$. A dominated convergence theorem concludes as above for the corresponding stochastic integral. At the end we have got weak convergence in L^1 of all terms so that :

$$\begin{aligned} (z + X_t^\epsilon)^{-1} &= \phi_t((z + X_0^\epsilon)^{-1}) + (1-\epsilon) \int_0^t \phi_{t-s} (\bar{\delta}((z + X_s^\epsilon)^{-1}) \# dS_s) \\ &+ ((1-\epsilon)^2 - 1) \sum_{i=1}^N \int_0^t ds \phi_{t-s} (m \circ 1 \otimes \tau \otimes 1((z + X_s^\epsilon)^{-1} \bar{\delta}_i(X_s^\epsilon) (z + X_s^\epsilon)^{-1} \bar{\delta}_i(X_s^\epsilon) (z + X_s^\epsilon)^{-1})). \end{aligned}$$

We now want to make ϵ tend to 0 after taking the trace to get the second statement. Note that in our context of section 2 where $\|\delta(x)\|_2 = \|\Delta^{1/2}(x)\|_2$, (5) gives :

$$(1 - (1-\epsilon)^2) \int_0^t \|\bar{\delta}(X_s^\epsilon)\|_2^2 ds = \|X_0\|_2^2 - \|X_t^\epsilon\|_2^2,$$

³remark that this second term is already known to exists; by boundedness in L^2 of the convergent $\delta \eta_\alpha(X_s^\epsilon)$, we get that $(z + \eta_\alpha(X_s^\epsilon))^{-1} \delta(\eta_\alpha(X_s^\epsilon))(z + \eta_\alpha(X_s^\epsilon))^{-1}$ is close in $\|\cdot\|_1$ of $(z + X_s^\epsilon)^{-1} \delta(\eta_\alpha(X_s^\epsilon))(z + X_s^\epsilon)^{-1}$, and finally, with convergence in L^2 of $(z + X_s^\epsilon)^{-1} \delta \eta_\alpha(X_s^\epsilon) (z + X_s^\epsilon)^{-1}$ to $(z + X_s^\epsilon)^{-1} \delta(X_s^\epsilon) (z + X_s^\epsilon)^{-1}$; we have thus obtained the convergence in L^1 , using that the two terms are known to be in L^2 and the sequence bounded in this space, you get the result.

Incidentally, this proves the statement that $\|X_0\|_2^2 = \|X_t\|_2^2$ in case (ii) of theorem 10 since we proved there convergence of X_t in L^2 and boundedness of $\|\bar{\delta}(X_s^\epsilon)\|_2$.

But (modulo extraction) the weak limit defining X_t gives $\|X_t\|_2 \leq \liminf \|X_t^\epsilon\|_2$ and thus

$$\limsup_{\epsilon \rightarrow 0} (1 - (1 - \epsilon)^2) \int_0^t \|\bar{\delta}(X_s^\epsilon)\|_2^2 ds \leq \|X_0\|_2^2 - \|X_t\|_2^2$$

And the last term is almost everywhere 0 under our assumption. Moreover, the trace of the second line of the equality of proposition 36 is bounded up to the cube of an inverse of $\Im(z)$ by this quantity, and thus we get almost everywhere (in t independent of z) equality of the Cauchy transforms of X_0 and X_t (using an argument as above to prove norm 2 convergence of X_t^ϵ to X_t on the a.e. set where $\|X_t\|_2 = \|X_0\|_2$), giving a.e. boundedness (and equality of von Neumann algebra norms). Now we can use the weak continuity proved in theorem 10 to extend boundedness everywhere.

Second, to prove that $X_t^\epsilon \in M$, consider $S_t^{(i,J)}$ $1 \leq i \leq N$, $J \in \{a, b\}$ a family of free Brownian motions, on which we extend δ by 0. We can always write $S_s^{(i)} = (1 - \epsilon)S_s^{(i,a)} + \sqrt{1 - (1 - \epsilon)^2}S_s^{(i,b)}$.

We have thus

$$X_t = \phi_t(X_0) + (1 - \epsilon) \int_0^t \phi_{t-s}(\delta(X_s) \# dS_s^{(a)}) + \sqrt{1 - (1 - \epsilon)^2} \int_0^t \phi_{t-s}(\delta(X_s) \# dS_s^{(b)}).$$

We want to prove that, if we apply E_a , the conditional expectation on the von Neumann algebra $M^{(a)}$ generated by M_0 and $S_s^{(a)}$, we get :

$$E_a(X_t) = \phi_t(X_0) + (1 - \epsilon) \int_0^t \phi_{t-s}(\delta(E_a(X_s)) \# dS_s^{(a)}),$$

which says nothing but by changing S_s in $S_s^{(a)}$, $E_a(X_t)$ is an instance of (the unique solution) X_t^ϵ getting the stated boundedness.

Since $E_a(\int_0^t \phi_{t-s}(\delta(\frac{1}{z+X_s}) \# dS_s^{(b)})) = 0$ is a consequence of freeness between $\{S_s^{(a)}\}$ and $\{S_s^{(b)}\}$, we just have to show several commutations of E_a with several operations, more precisely : $E_a \phi_t = \phi_t E_a$, $E_a(\cdot \# (S_t^{(a)} - S_s^{(a)})) = (E_a \bar{\otimes} E_a(\cdot)) \# (S_t^{(a)} - S_s^{(a)})$ on $L^2(M_s)$ and $E_a \bar{\otimes} E_a \circ \bar{\delta} = \bar{\delta} \circ E_a$. With that and obvious lemmas about stochastic integrals, we will have what we want. The first equation is nothing but an instance of the preservation (contained in the preliminaries of part 2.1 with this new case of zero extension) by Δ of $M^{(a)}$ (and characterization of conditional expectation). The second is proved in using also the characterization of conditional expectation once noted that we can use instead of someone in $L^2(M^{(a)})$, someone in $L^2(M_s^{(a)} \otimes M_s^{(a)}) \# (S_t^{(a)} - S_s^{(a)})$ by orthogonality. The third one is verified by using the fact that $\delta^* : L^2(M^{(a)} \otimes M^{(a)}) \rightarrow L^2(M^{(a)})$ (and characterization of conditional expectation). \square

3.3. Stationarity.

Proposition 37. *Let us call $\Phi_t : X_0 \in M_0 \cap D(\bar{\delta}) \mapsto X_t \in M_t$ the previous ultramild solution of theorem 10 assuming $\|X_t\|_2 = \|X_0\|_2$ a.e (modulo the boundedness of the previous proposition for the range). Then, $\Phi_t(X_0 Y_0) = \Phi_t(X_0) \Phi_t(Y_0)$ if $X_0, Y_0 \in D(\bar{\delta}) \cap M_0$.*

Proof. Since $\Phi_t(X^*) = \Phi_t(X)^*$, $\Phi_t(1) = 1$ and τ is faithful, $D(\bar{\delta}) \cap M_0$ a *-algebra, it suffices to prove that for any $X_0, Y_0, Z_0, T_0 \in D(\bar{\delta}) \cap M_0$
 $\tau(\Phi_t(X_0)\Phi_t(Y_0)\Phi_t(Z_0)\Phi_t(T_0)) = \tau(X_0Y_0Z_0T_0)$. For notational convenience we prove only the case $Z_0 = T_0 = 1$ (even if this case is also a direct consequence of the assumed isometry by polarization), the general similar case being left to the reader.

Let also $\Phi_t^\epsilon : X_0 \in M_0 \cap D(\bar{\delta}) \mapsto X_t^\epsilon \in M_t$.

Apply Ito's formula (assumptions of proposition 35) to $\eta_\alpha(X_t^\epsilon)$, and $\eta_\alpha(Y_t^\epsilon)$ (using they are valued in M) :

$$\begin{aligned} \eta_\alpha(X_t^\epsilon)\eta_\alpha(Y_t^\epsilon) &= \eta_\alpha(X_0^\epsilon)\eta_\alpha(Y_0^\epsilon) + (1-\epsilon) \int_0^t \eta_\alpha^\otimes(\bar{\delta}(X_s^\epsilon))\eta_\alpha(Y_s^\epsilon) + \eta_\alpha(X_s^\epsilon)\eta_\alpha^\otimes(\bar{\delta}(Y_s^\epsilon))\#dS_s \\ &\quad - \frac{1}{2} \int_0^t \eta_\alpha(X_s)\Delta\eta_\alpha(Y_s) + \Delta\eta_\alpha(X_s)\eta_\alpha(Y_s)ds \\ &\quad + (1-\epsilon)^2 \sum_{i=1}^N \int_0^t m \circ 1 \otimes \tau \circ m \otimes 1(\eta_\alpha^\otimes(\bar{\delta}_i(X_s^\epsilon)) \otimes \eta_\alpha^\otimes(\bar{\delta}_i(Y_s^\epsilon)))ds. \end{aligned}$$

We can now use lemma 13 to get :

$$\begin{aligned} \eta_\alpha(X_t^\epsilon)\eta_\alpha(Y_t^\epsilon) &= \eta_\alpha(X_0^\epsilon)\eta_\alpha(Y_0^\epsilon) \\ &\quad + (1-\epsilon) \int_0^t \bar{\delta}(\eta_\alpha(X_s^\epsilon)\eta_\alpha(Y_s^\epsilon))\#dS_s - \frac{1}{2} \int_0^t \Delta^1(\eta_\alpha(X_t^\epsilon)\eta_\alpha(Y_t^\epsilon))ds \\ &\quad + ((1-\epsilon)^2 - 1) \sum_{i=1}^N \int_0^t m \circ 1 \otimes \tau \circ m \otimes 1(\bar{\delta}_i(\eta_\alpha(X_s^\epsilon)) \otimes \bar{\delta}_i(\eta_\alpha(Y_s^\epsilon)))ds \\ &\quad + (1-\epsilon) \int_0^t (\eta_\alpha^\otimes(\bar{\delta}(X_s^\epsilon)) - \bar{\delta}(\eta_\alpha(X_s^\epsilon)))\eta_\alpha(Y_s^\epsilon)\#dS_s \\ &\quad + (1-\epsilon) \int_0^t (\eta_\alpha(X_s^\epsilon)(\eta_\alpha^\otimes(\bar{\delta}(Y_s^\epsilon)) - \bar{\delta}(\eta_\alpha(Y_s^\epsilon)))\#dS_s \\ &\quad + (1-\epsilon)^2 \sum_{i=1}^N \int_0^t m \circ 1 \otimes \tau \circ m \otimes 1((\eta_\alpha^\otimes(\bar{\delta}_i(X_s^\epsilon)) - \bar{\delta}_i(\eta_\alpha(X_s^\epsilon))) \otimes \eta_\alpha^\otimes(\bar{\delta}_i(Y_s^\epsilon)))ds \\ &\quad + (1-\epsilon)^2 \sum_{i=1}^N \int_0^t m \circ 1 \otimes \tau \circ m \otimes 1(\bar{\delta}_i(\eta_\alpha(X_s^\epsilon)) \otimes (\eta_\alpha^\otimes(\bar{\delta}_i(Y_s^\epsilon)) - \bar{\delta}_i(\eta_\alpha(Y_s^\epsilon))))ds. \end{aligned}$$

Using once again the trick of proposition 7 to pass to something which looks like a mild solution, then we can take the limit $\alpha \rightarrow \infty$ as in Proposition 36 and finally we get (using that $\Phi_t^\epsilon(X_0Y_0)$ is a mild solution since $X_0Y_0 \in D(\bar{\delta}) \cap M$) :

$$\begin{aligned} \Phi_t^\epsilon(X_0)\Phi_t^\epsilon(Y_0) - \Phi_t^\epsilon(X_0Y_0) &= (1-\epsilon) \int_0^t \phi_{t-s}(\bar{\delta}(\Phi_s^\epsilon(X_0)\Phi_s^\epsilon(Y_0) - \Phi_s^\epsilon(X_0Y_0))\#dS_s) \\ &\quad + ((1-\epsilon)^2 - 1) \sum_{i=1}^N \int_0^t \phi_{t-s} (m \circ 1 \otimes \tau \circ m \otimes 1(\bar{\delta}_i(\Phi_s^\epsilon(X_0)) \otimes \bar{\delta}_i(\Phi_s^\epsilon(Y_0)))) ds. \end{aligned}$$

Since $\Phi_t^c(X_0)$ converges in $\|\cdot\|_2$ -norm to $\Phi_t(X_0)$ (a.e) (using again assumption on $\|\cdot\|_2$ -isometry), we can show that, after taking the trace, this equation converges to the wanted relation $\tau(\Phi_t(X_0)\Phi_t(Y_0)) = \tau(X_0Y_0)$ since also the last term goes to zero as in proposition 36.

□

4. APPLICATIONS

4.1. Free Difference Quotient.

Corollary 38. *Assume assumption 1 and $X_1, \dots, X_n \in D(\Delta) \cap M_0$. Then for any $t \geq 0$ there exists an embedding $\Phi_t : M_0 = W^*(X_1, \dots, X_n) \rightarrow M_0 * L(F(\infty))$ and $S_1, \dots, S_N \in L(F(\infty))$ a free $(0, 1)$ -semicircular family, free from M_0 and such that :*

$$\|\Phi_t(X_j) - X_j - \sqrt{t} \sum_{i=1}^N \bar{\partial}_i(X_j) \# S_i\|_2 \leq c_j t,$$

for a fixed constant

$$c_j^2 = \frac{1}{4} \|\Delta(X_j)\|_2^2 + \frac{1}{2} \left(\|\Delta^{\otimes 1/2}(\delta(X_j))\|_2^2 + \frac{\pi}{4} \|\Delta(X_j)\|_2 \|\Delta^{\otimes 1/2}(\delta(X_j))\|_2 \right).$$

Moreover, $\Phi_t(X_j) \in W^*(X_1, \dots, X_n, S_1, \dots, S_N, \{S'_j\}_{j=0}^\infty)$ where $\{S'_j\}_{j=0}^\infty$ is a free semicircular family free with $\{X_1, \dots, X_n, S_1, \dots, S_N\}$.

As a consequence, if we define $c^2 = \sum c_j^2$, we have the following inequality for the Wasserstein-Biane-Voiculescu distance ([4]) :

$$d_W(\mu_{X_1, \dots, X_n}, \mu_{X_1 + \sqrt{t}\delta(X_1) \# S, \dots, X_n + \sqrt{t}\delta(X_n) \# S}) \leq c t.$$

As another consequence, using [46, Theorem 4], any R^ω -embeddable von Neumann algebra generated by X_1, \dots, X_n with Lipschitz conjugate variable have $\delta_0(X_1, \dots, X_n) = n$.

Remark 39. This result is analogous to proposition 2 in [46], and to an inequality in [4], but the latter is for the free difference quotient for $n = 1$ with only finite Fisher information and the former for any derivation assuming $\partial(X_j)$ and $\partial^* \partial(X_j)$ can be written in terms of non-commutative power series and for a general n . Compared to these results, our result can be applied for a general n but for “almost coassociative” derivations, and for the free difference quotient with only the assumption Lipschitz conjugate variable (i.e. $\bar{\partial} \partial_j^* 1 \otimes 1 \in (M \overline{\otimes} M^{op})^n$, which corresponds to Lipschitz conjugate variable in the $n = 1$ case, cf. also [57] for a more general justification of this terminology.) Note also that in that case the constant is expressed in terms of free Fisher information $\Phi^*(X_1, \dots, X_n) = \sum_i \|\Delta(X_i)\|_2^2$, it becomes the expected $c = \Phi^*(X_1, \dots, X_n)^{1/2}/2$, so that for instance if X_1, \dots, X_n is such that the associated Ornstein-Ühlenbeck process $Y_i(t) = e^{-t/2} X_i + (1 - e^{-t})^{1/2} S_i$ satisfy $X_1(t), \dots, X_n(t)$ have Lipschitz conjugate variable (in the above sense for all $t > 0$, which is by no means a trivial assumption) then the argument of [4] gives the corresponding free Talagrand transportation cost inequality :

$$d_W((X_1, \dots, X_n), (S_1, \dots, S_N)) \leq$$

$$\sqrt{2} \left(\chi^*(S_1, \dots, S_N) - \chi^*(X_1, \dots, X_n) - \frac{n}{2} + \frac{1}{2} \sum_{i=1}^N \tau(X_i^2) \right)^{1/2}$$

We prove in [12] this result in full generality using another way of solving stochastic differential equations.

We give a concrete non-trivial example of Lipschitz conjugate variable in subsection 4.3.

Sketch of Proof. For the reader's convenience we outline how this follows from the beginning of the paper. Using Assumption 1, theorem 28 gives the conditions to apply theorem 10 (ii) with $\omega = 0$. Then $\Phi_t(X) = X_t$ is given by the mild solution of the SDE given by (ii) and the stated inequality is the one coming from (i) in theorem 10 (the inequality on Wasserstein distance is then an obvious consequence, note that $\delta(X_0) \in D((\Delta \otimes 1 + 1 \otimes \Delta)^{1/2})$ follows from lemma 31 (v)). The fact that Φ_t gives a *-homomorphism comes from Proposition 37. Since it preserves the trace by the SDE it satisfies, we can extend it at the von Neumann algebraic level. (S_i, S'_j) are produced from the free Brownian motion of the SDE). Assumption 1 is true in case of Lipschitz conjugate variable as follows. First, the free difference quotient being coassociative, (a) is true with $C = 0$. (a') (with $p = \infty$ is true by the definition on non-commutative polynomial and because lipschitz conjugate variables imply the conjugate variables are in M (using e.g. the equality (1) in [11]). (b) is valid (directly via Lipschitz conjugate variable assumption) for $p = \infty$ in the context when $Comp_{\Pi^2_{\infty, \infty}}(R_{\infty, \epsilon}, 2 + \epsilon, 2, \infty)$ holds (cf remark 21) with $R_{\infty, \epsilon} = (2 + \epsilon, 2 + \epsilon, 1 + \epsilon/2, 1 + \epsilon/2, 1 + \epsilon/2)$. with thus $\tilde{\rho} = 2 + \epsilon, \tilde{\sigma} = 1 + \epsilon/2$.

Since \mathcal{B} contains non-commutative polynomials (c) is obvious.

As stated, the equality on microstate free entropy dimension then comes from [46, Theorem 4]. \square

4.2. Preliminaries and relations of three natural derivations on q -Gaussian factors. Our goal is to study three derivations on q -Gaussian factors : the free difference quotient, the commutator with right creation operators and the one giving the number operator as generator of the associated Dirichlet form. Especially, we want to find values of q 's for which they can be seen as closed derivations with value in the coarse correspondence, and equivalent in the sense of definition 11.

We will use this preliminaries to apply our results in the next two subsections.

4.2.1. Preliminaries on q -Gaussian factors. We recall the construction of q -Gaussian variables given by Bożejko and Speicher in [7].

Let $N < \infty$ be an integer, $\mathcal{H} = \mathbb{R}^N$, $\mathcal{H}_C = \mathbb{C}^N$ its complexification, and $-1 < q < 1$. Consider the vector space

$$F_{alg}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \mathcal{H}_C^{\otimes n}$$

(algebraic direct sum and tensor products). This vector space is endowed with a positive definite inner product given by

$$\begin{aligned} \langle \xi_1 \otimes \cdots \otimes \xi_n, \zeta_1 \otimes \cdots \otimes \zeta_m \rangle_q &= \delta_{n=m} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{j=1}^n \langle \xi_j, \zeta_{\pi(j)} \rangle \\ &= \delta_{n=m} \langle \xi_1 \otimes \cdots \otimes \xi_n, P_q^{(n)} \zeta_1 \otimes \cdots \otimes \zeta_n \rangle_0, \end{aligned}$$

where $i(\pi) = \#\{(i, j) : i < j \text{ and } \pi(i) > \pi(j)\}$, and $P_q^{(n)} = \sum_{\pi \in S_n} q^{i(\pi)} \pi$ where π acts via $\pi^{-1}(\zeta_1 \otimes \cdots \otimes \zeta_n) = \zeta_{\pi(1)} \otimes \cdots \otimes \zeta_{\pi(n)}$. Denote by $F_q(\mathcal{H})$ the completion of $F_{alg}(\mathcal{H})$ with respect to this inner product.

For $h \in \mathcal{H}$, define $\ell(h) : F_q(\mathcal{H}) \rightarrow F_q(\mathcal{H})$ by extending continuously the map

$$\begin{aligned} \ell(h)h_1 \otimes \cdots \otimes h_n &= h \otimes h_1 \otimes \cdots \otimes h_n, \\ \ell(h)\Omega &= h. \end{aligned}$$

The adjoint is given by

$$\begin{aligned} \ell^*(h)h_1 \otimes \cdots \otimes h_n &= \sum_{k=1}^n q^{k-1} \langle h_k, h \rangle h_1 \otimes \cdots \otimes \hat{h}_k \otimes \cdots \otimes h_n, \\ \ell^*(h)\Omega &= 0, \end{aligned}$$

where $\hat{\cdot}$ denotes omission. $\omega(h) = \ell(h) + \ell^*(h)$ are q -Gaussian variables. $\Gamma_q(\mathcal{H})$ is the von Neumann algebra generated by $\omega(h)$ $h \in \mathcal{H}$, acting as bounded operators on $F_q(\mathcal{H})$. We use on it the faithful trace $\tau_q(X) = \langle X\Omega, \Omega \rangle$. It is well-known that $L^2(\Gamma_q(\mathcal{H}), \tau_q) \simeq F_q(\mathcal{H})$. For $\xi \in F_{alg}(\mathcal{H})$ we write $\psi(\xi)$ the element in $\Gamma_q(\mathcal{H})$ such that $\psi(\xi)\Omega = \xi$, associated to this identification (since it is easy to see that $F_{alg}(\mathcal{H}) \subset F_q(\mathcal{H})$ is identified with a subspace of $\Gamma_q(\mathcal{H}) \subset L^2(\Gamma_q(\mathcal{H}), \tau_q)$ corresponding to polynomials in $\omega(h)$'s).

Consider also $r(h)$ given by

$$\begin{aligned} r(h)h_1 \otimes \cdots \otimes h_n &= h_1 \otimes \cdots \otimes h_n \otimes h \\ r(h)\Omega &= h. \end{aligned}$$

Finally, let $P_n : F_q(\mathcal{H}) \rightarrow F_q(\mathcal{H})$ be the orthogonal projection onto tensors of rank n . Let $\Xi_q = \sum_{n \geq 0} q^n P_n$. It is obvious that Ξ_q is an Hilbert-Schmidt operator as soon as $q^2 N < 1$. We also introduce a natural finite rank approximation $\Xi_q^Q = \sum_{n=0}^Q q^n P_n$.

4.2.2. Three natural derivations on q -Gaussian factors. The following lemma is proven in [44] (and stated exactly in that way in [46, Lemma 5], and (iii) is proved using also [45, Theorem 1]).

Fix an orthonormal basis $\{h_i\}_{i=1}^N \subset \mathbb{R}^N$ and let $X_i = \omega(h_i)$. Thus $\Gamma_q(\mathcal{H}) = W^*(X_1, \dots, X_N)$, $N = \dim \mathcal{H}_{\mathbb{R}}$. We may also write for $\underline{i} = (i_1, \dots, i_n) \in N^n$ $\psi_{\underline{i}} = \psi(h_{i_1} \otimes \dots \otimes h_{i_n})$. Finally, for a von Neumann algebra M , M^{op} will be as usual the opposite algebra. Later, I will consider $M = \Gamma_q(\mathcal{H})$.

Lemma 40. [44] For $j = 1, \dots, N$, $q^2 N < 1$, let $\partial_j^{(q)} : \mathbb{C}\langle X_1, \dots, X_N \rangle \rightarrow HS$ be the derivation given by $\partial_j^{(q)}(X_i) = \delta_{i=j} \Xi_q = [X_i, r(h_j)] = [r(h_j)^*, X_i]$. Let $\partial : \mathbb{C}\langle X_1, \dots, X_N \rangle \rightarrow HS^N$ be given by $\partial^{(q)} = \partial_1^{(q)} \oplus \cdots \oplus \partial_N^{(q)}$ and regard ∂ as an unbounded operator densely defined on $L^2(\Gamma_q(\mathcal{H}))$. Then:

(i) $\partial^{(q)}$ is closable.

(ii) If we denote by Z_j the vector $0 \oplus \cdots \oplus P_{\Omega} \oplus \cdots \oplus 0 \in HS^N$ (nonzero entry in j -th place, P_{Ω} is the orthogonal projection onto $\mathbb{C}\Omega \in F_q(\mathcal{H})$), then Z_j is in the domain of ∂^* and $\partial^{(q)*}(Z_j) = h_j$.

(iii) $1 \otimes \tau(\partial_j^{(q)}(X)) = \partial_j^{(q)}(X)\Omega = r(h_j)^*(X.\Omega)$ (in the first equality we identify isometrically HS with $L^2(\Gamma_q(\mathcal{H}) \otimes \Gamma_q(\mathcal{H})^{op})$ as usual via $a \otimes b$ with the rank one operator $a\tau(b.)$)

Let us recall the following crucial result of Bożejko ([5]) giving an Haagerup like inequality for q -Gaussian variables.

Theorem 41. (*Haagerup-Bożejko Inequality* [5]) *If $C_q^{-1} = \prod_{m=1}^{\infty} (1 - q^m)$ then for any $\xi \in \mathcal{H}^{\otimes n} \subset F_{alg}(\mathcal{H})$:*

$$\|\psi(\xi)\|_{L^2(\Gamma_q(\mathcal{H}), \tau_q)} \leq \|\psi(\xi)\|_{\Gamma_q(\mathcal{H})} \leq C_{|q|}^{3/2} (n+1) \|\psi(\xi)\|_{L^2(\Gamma_q(\mathcal{H}), \tau_q)}.$$

Moreover, for any $\eta \in \mathcal{H}^{\otimes n} \otimes \mathcal{H}^{\otimes m} \subset F_{alg}(\mathcal{H}) \otimes_{alg} F_{alg}(\mathcal{H})$ (ϵ either *op* or *nothing*)

$$\|\psi \otimes \psi(\eta)\|_{\Gamma_q(\mathcal{H}) \overline{\otimes} \Gamma_q(\mathcal{H})^\epsilon} \leq C_{|q|}^3 (n+1)(m+1) \|\psi \otimes \psi(\eta)\|_{L^2(\Gamma_q(\mathcal{H}) \otimes \Gamma_q(\mathcal{H})^{op}, \tau_q \otimes \tau_q)}.$$

A short proof of the first part can be found in [30] (basically variant of [5] without writing the computations), the argument obviously giving the second part too. Alternatively, we can apply to $u_i = \psi$ (and a variant with right multiplication in the case $\epsilon = op$) of the fact that if $u_i : H_i \rightarrow B(K_i)$ are bounded maps from Hilbert spaces to bounded maps on a Hilbert space (nothing but trilinear forms on Hilbert spaces), then their tensor product $u_1 \otimes u_2$ is bounded from $H_1 \otimes H_2$ to $B(K_1 \otimes K_2)$ with $\|u_1 \otimes u_2\| \leq \|u_1\| \|u_2\|$.

From now on ψ may not be written explicitly, no more than identifications between $L^2(\Gamma_q(\mathcal{H}) \otimes \Gamma_q(\mathcal{H})^{op})$ and Hilbert-Schmidt operators (following section 2, but here the adjoint being the one coming from $\Gamma_q(\mathcal{H}) \otimes \Gamma_q(\mathcal{H})^{op}$ if not specified explicitly).

As a consequence, for any $\xi \in \bigoplus_{p \leq n} \mathcal{H}^{\otimes p}$ of component ξ_p , we also have by Cauchy-Schwarz :

$$\begin{aligned} \|\psi(\xi)\|_{\Gamma_q(\mathcal{H})} &\leq C_{|q|}^{3/2} \sum (p+1) \|\psi(\xi_p)\|_{L^2(\Gamma_q(\mathcal{H}), \tau_q)} \\ (14) \quad &\leq C_{|q|}^{3/2} (n+1)^{3/2} \left(\sum \|\psi(\xi_p)\|_{L^2(\Gamma_q(\mathcal{H}), \tau_q)}^2 \right)^{1/2} \\ &= C_{|q|}^{3/2} (n+1)^{3/2} \|\psi(\xi)\|_{L^2(\Gamma_q(\mathcal{H}), \tau_q)} \end{aligned}$$

Likewise, for any $\eta \in \bigoplus_{p+q \leq n} \mathcal{H}^{\otimes p} \otimes \mathcal{H}^{\otimes q}$ we also have :

$$(15) \quad \|\psi(\eta)\|_{\Gamma_q(\mathcal{H}) \overline{\otimes} \Gamma_q(\mathcal{H})^{op}} \leq C_{|q|}^3 (n+1)^3 \|\psi(\eta)\|_{L^2(\Gamma_q(\mathcal{H}) \otimes \Gamma_q(\mathcal{H})^{op}, \tau_q \otimes \tau_q)}$$

In order to state the next result, let us fix several notation about tensor products (similar to those of [51] 3.1). M is a given finite II_1 factor with faithful normal trace τ . $M \hat{\otimes} M^{op}$ is the projective tensor product of M with its opposite algebra, with the corresponding $*$ -Banach algebra structure. Let $\alpha : M \hat{\otimes} M^{op} \rightarrow B(M)$ be the contractive homomorphism given by $\alpha(a \otimes b) = L_a R_b$, where L_a and R_b are respectively the left and right multiplication operators by a and b . We will denote $LR(M)$ the algebra $\alpha(M \hat{\otimes} M^{op})$. It is easily seen that $\|\alpha(x)m\|_p \leq \|x\|_{M \hat{\otimes} M^{op}} \|m\|_p$, for $1 \leq p \leq \infty$ so that $LR(M)$ acts boundedly on $L^p(M, \tau)$ (the completion of M with respect to $\|x\|_p = \tau(|x|^p)^{1/p}$). Consistently with previous notations, we will write $x \# m$ any of those actions (and several others we are about to discuss). For $p = 2$ this gives a map $\beta : LR(M) \rightarrow C^*(M, M')$ where M , and M' are with respect to the standard form of M on $L^2(M)$. Further we have a $*$ -homomorphism $\gamma : C^*(M, M') \rightarrow M \overline{\otimes} M^{op}$ with value in the von Neumann algebra tensor product given by the general C^* tensor product theory. We will of course see $M \overline{\otimes} M^{op}$ as a II_1 factor with canonical trace $\tau \otimes \tau$. Finally, we will write $\#$ any "side multiplication" when defined. For instance, $a \otimes b \# a' \otimes b' \# a'' \otimes b'' = aa'a'' \otimes b''b'b$

so that $\#$ may be in this case multiplication in $M \overline{\otimes} M^{op}$, or any of its induced actions on $L^2(M \otimes M^{op})$. More generally, for $i \in [1, p-1]$, $a_i, b_j \in M$, we write

$$(a_1 \otimes a_2 \otimes \dots \otimes a_p) \#_i (b_1 \otimes \dots \otimes b_n) = a_1 \otimes \dots \otimes a_i b_1 \otimes b_2 \otimes \dots \otimes b_n a_{i+1} \otimes \dots \otimes a_p$$

(if $p = 2$, $\#_1 = \#$), and likewise the corresponding extension for instance $M^{\overline{\otimes} i} \hat{\otimes} M^{\overline{\otimes} p-i} \times M^{\overline{\otimes} n} \rightarrow M^{\overline{\otimes} n+p-2}$ (or any analogues containing M^{op} the multiplication being then consistently defined to get what expected above in M as if there where everywhere M , for instance if $a \otimes a', c \otimes c' \in M \otimes M^{op}$, $b \otimes b' \otimes b'' \in (M \otimes M) \hat{\otimes} M^{op}$ we have $(a \otimes a') \#_1 ((b \otimes b' \otimes b'') \#_2 (c \otimes c')) = ((a \otimes a') \#_1 (b \otimes b' \otimes b'')) \#_2 (c \otimes c') = (ab \otimes b'c \otimes c'b''a') \in M \otimes M \otimes M^{op}$ (all multiplications written in M , if we were more consistent with M^{op} we would have written $a'b''c'$). However, we won't use this notation if $c \otimes c'$ is thought of in $M \otimes M$, but everything would be the same if also $b \otimes b' \otimes b'' \in (M \otimes M^{op}) \hat{\otimes} M^{op}$ except for the value in this space in $M \otimes M^{op} \otimes M^{op}$).

We will often use the following assumption and give an easy sufficient condition deduced from Bozejko inequality in the next corollary.

Assumption $I_q : q\sqrt{N} < 1$ and Ξ_q is invertible in $M \overline{\otimes} M^{op}$

Even if we will scarcely use it, for $R > 1$ and a non-commutative power series (of radius of convergence larger than ρ with value in a tensor product) $F(Y_1, \dots, Y_n) = \sum a_{i_1, \dots, i_n, p} Y_{i_1} \dots Y_{i_p} \otimes Y_{i_{p+1}} \dots Y_{i_n}$ we write the usual norm $\|F\|_R = \sum |a_{i_1, \dots, i_n, p}| R^n$. We will use the same notation with less or more tensors in the space of value.

Corollary 42. *When the right hand side in the inequalities bellow is finite, Ξ_q comes from an element in $M \hat{\otimes} M^{op}$, and respectively $M \overline{\otimes} M^{op}$ via $\iota \gamma \beta \alpha$ or $\iota : M \overline{\otimes} M^{op} \rightarrow L^2(M \overline{\otimes} M^{op})$ and with obvious notations:*

$$\|\Xi_q - 1 \otimes 1\|_{M \hat{\otimes} M^{op}} \leq (C_{|q|})^3 \left[\frac{4|q|N}{1 - |q|N} + \frac{5(|q|N)^2}{(1 - |q|N)^2} + \frac{2(|q|N)^3}{(1 - |q|N)^3} \right] =: \nu(q, N)$$

$$\|\Xi_q - 1 \otimes 1\|_{M \overline{\otimes} M^{op}} \leq (C_{|q|})^3 \left[\frac{4|q|\sqrt{N}}{1 - |q|\sqrt{N}} + \frac{5(|q|\sqrt{N})^2}{(1 - |q|\sqrt{N})^2} + \frac{2(|q|\sqrt{N})^3}{(1 - |q|\sqrt{N})^3} \right] =: \rho(q, N)$$

Epecially ($N \geq 2$) if q is such that $\nu(q, N) < 1$, e.g. for $|q|N \leq 0.13$, then Ξ_q is invertible in $M \hat{\otimes} M^{op}$ (resp. if q is such that $\rho(q, N) < 1$ e.g. when $|q|\sqrt{N} \leq 0.13$ then I_q holds) Moreover, if $q\sqrt{N} < 1$, $\|\Xi_q^Q - \Xi_q\|_{M \overline{\otimes} M^{op}} \rightarrow_{Q \rightarrow \infty} 0$ and $\Xi_q \in C^(X_1 \otimes 1, 1 \otimes X_1, \dots, 1 \otimes X_N) \subset M \overline{\otimes} M^{op}$ is positive so that $\Xi_q^{1/2}$ is well defined.*

Moreover, if $\epsilon > 0$ and $(4(2 + \epsilon)^2 N + 2)|q| < 1$ there exists a non-commutative power series $\Xi_q(Y_1, \dots, Y_N)$ with radius of convergence greater than $R = (1 + \epsilon) \frac{2}{1 - |q|} > \|X_i\|$ such that $\Xi_q(X_1, \dots, X_N) = \Xi_q$, and

$$\|\Xi_q - 1 \otimes 1\|_R \leq \frac{4(2 + \epsilon)^2 N |q|}{1 - (2 + 4(2 + \epsilon)^2 N) |q|},$$

and likewise,

$$\max(\|\partial_i \otimes 1(\Xi_q)\|_R, \|1 \otimes \partial_i(\Xi_q)\|_R) \leq \frac{4(2 + \epsilon)^2 N |q| (1 - 2|q|)}{(1 - (2 + 4(2 + \epsilon)^2 N) |q|)^2}$$

Proof. Since P_n can be seen as a finite rank operator written as $\sum_{\xi} \xi \otimes \xi^*$ with the usual identification (the sum running over an orthonormal basis of $\mathcal{H}^{\otimes n}$), the previous theorem gives : $\|P_n\|_{M \otimes M^{op}} \leq \sum_{\xi} \|\xi\|^2 \leq C_{|q|}^3 (n+1)^2 \sum_{\xi} \|\xi\|_2^2 = C_{|q|}^3 (n+1)^2 N^n$. The inequality follows from a standard computation.

Likewise

$$\|\Xi_{q-1} \otimes 1\|_{M \otimes M^{op}} \leq \sum_{n \geq 1} q^n \left\| \sum_{\xi} \xi \otimes \xi^* \right\| \leq C_{|q|}^3 \sum_{n \geq 1} q^n (n+1)^2 \left\| \sum_{\xi} \xi \otimes \xi^* \right\|_2 = C_{|q|}^3 \sum_{n \geq 1} q^n (n+1)^2 N^{n/2}.$$

Let us call $f(qN)$ the term in brackets in the right hand side of the inequality. to get $f(qN) < C_{|q|}^{-3}$, it suffices to have $f(qN) < (1 + |q|)^3 \prod_{m=1}^{\infty} (1 - |q|^m)^3 / (1 + |q|^m)^3 = (\sum_{n \in \mathbb{Z}} (-1)^n |q|^{n^2})^3$, and again keeping only the smallest order it suffices to have $f(|q|N) < (1 - |q| - 2|q|^2)^3$ and solving numerically $f(qN) < (1 - |q|N/2 - |q|^2 N^2/2)^3$ (sufficient since $N \geq 2$) one gets $|q|N < 0.1386...$

For the last statement, we only improve an estimate in [46]. We write $p_{\underline{i}}$ the polynomials giving, by evaluation on X_1, \dots, X_N , the orthonormalization of $\psi_{\underline{i}}$ defined in lemma 8 in [46]. More specifically, we consider Γ_n the Gramm matrix of q -scalar products in the space of tensors of length n given (for $|\underline{j}| = |\underline{l}| = n$) by : $(\Gamma_n)_{(\underline{j}, \underline{l})} = \langle \psi_{j_1, \dots, j_n}, \psi_{l_1, \dots, l_n} \rangle_q$, this is an $N^n \times N^n$ matrix known to be positive and invertible (with real coefficients), and we consider $B = \Gamma^{-1/2}$. Note that by definition it is given by the image of the element of $P_q^{(n)} = \sum_{\pi \in S_n} q^{i(\pi)} \pi$ in the algebra of the symmetric group S_n by the obvious representation $\pi_{q, N, n}$ of S_n on (the formal basis of the \mathbb{C}^{N^n}) $\xi_{\underline{l}}$ and it is known from [16] and [59] a formula for $P_q^{(n)-1}$ given by the inductive relation $P_q^{(n)} = \pi_{n-1, n}(P_q^{(n-1)}) M_n, \pi_{n-1, n}$ the usual embedding of S_{n-1} in S_n with image leaving 1 invariant and $M_n = \sum_{k=1}^n q^{k-1} (1 \rightarrow k)$ (with the notation $(k \rightarrow l)$ the cycle sending $k+i$ to $k+i+1$ for $0 \leq i \leq l-k-1$ and sending l to k) via $M_n^{-1} = \prod_{j=n-1}^1 (1 - q^j (1 \rightarrow j+1)) \prod_{j=n-2}^0 (1 - q^{n-j} (2 \rightarrow n-j))^{-1}$. We will use it through $B^2 = \pi_{q, N, n}(P_q^{(n)-1})$. We also write $\psi_{\underline{j}}(Y_1, \dots, Y_N)$ the non-commutative polynomial defined inductively by ($\psi_{\epsilon} = 1$ for the empty word ϵ):

$$\psi_{i_1, \dots, i_n} = Y_{i_1} \psi_{i_2, \dots, i_n} - \sum_{j \geq 2} q^{j-2} \delta_{i_1 = i_j} \psi_{i_2, \dots, \hat{i}_j, \dots, i_n}.$$

As in the proof of proposition 2.7 in [6], we use the following identity for $\psi_{\underline{i}} = \psi_{\underline{i}}(X_1, \dots, X_N)$ for $\psi_{\underline{i}}$ introduced before.

Then by definition, $p_{\underline{i}}(Y_1, \dots, Y_N) = \sum_{\underline{j}, |\underline{j}|=n} B_{\underline{i}, \underline{j}} \psi_{\underline{j}}(Y_1, \dots, Y_N)$ so that (as checked in lemma 8 in [46]) $\{p_{\underline{i}}(X_1, \dots, X_N) \Omega\}_{|\underline{i}|=n}$ is obviously an orthonormal basis of $\mathcal{H}^{\otimes n}$.

$\Xi_q(Y_1, \dots, Y_N) = \sum_n q^n \sum_{\underline{i}} p_{\underline{i}}(Y_1, \dots, Y_N) \otimes p_{\underline{i}}^*(Y_1, \dots, Y_N)$ will be the power series we are looking for once proved an estimate on its norm. It suffices to bound (using symmetry of the matrix B) :

$$\begin{aligned} \left\| \sum_{\underline{i}} p_{\underline{i}}(Y_1, \dots, Y_N) \otimes p_{\underline{i}}^*(Y_1, \dots, Y_N) \right\|_R &= \left\| \sum_{\underline{i}, \underline{j}, \underline{l}} B_{\underline{i}, \underline{j}} \psi_{\underline{j}}(Y_1, \dots, Y_N) \otimes B_{\underline{i}, \underline{l}} \psi_{\underline{l}}^*(Y_1, \dots, Y_N) \right\|_R \\ &\leq \sum_{\underline{l}} \left\| \sum_{\underline{j}} B_{\underline{l}, \underline{j}}^2 \psi_{\underline{j}}(Y_1, \dots, Y_N) \right\|_R \left\| \psi_{\underline{l}}^*(Y_1, \dots, Y_N) \right\|_R \end{aligned}$$

Now using the the expression for B^2 expanded from the inverse coming from the action of the symmetric group algebra, it involves only $\|\psi_{\underline{\sigma(j)}}(Y_1, \dots, Y_N)\|_\rho$ and from the bound in [16] lemma 4.1, one gets

$$\|\sum_{\underline{j}} B_{\underline{l}, \underline{j}}^2 \psi_{\underline{j}}(Y_1, \dots, Y_N)\|_\rho \leq \left((1 - |q|) \prod_{k=1}^{\infty} \frac{1 + |q|^k}{1 - |q|^k} \right)^n \sup_{\sigma \in S_n} \|\psi_{\sigma(\underline{j})}(Y_1, \dots, Y_N)\|_R.$$

Finally, lemma 6 in [46] shows any coefficient of a monomial of degree $k \leq n$ in $\psi_{\underline{j}}$ is at most $2^{n-k} (\frac{1}{1-|q|})^{n-k}$ and its very definition shows that only the at most $\binom{n}{k}$ sub-monomials of $X_{j_1} \dots X_{j_n}$ can occur so that $\|\psi_{\underline{j}}(Y_1, \dots, Y_N)\|_\rho \leq \sum_{k=0}^n \binom{n}{k} (\frac{2}{1-|q|})^{n-k} \rho^k = (\rho + \frac{2}{1-|q|})^n$. Likewise, we have $\|\partial_i \psi_{\underline{j}}(Y_1, \dots, Y_N)\|_\rho \leq \sum_{k=0}^n k \binom{n}{k} (\frac{2}{1-|q|})^{n-k} \rho^{k-1} = n(\rho + \frac{2}{1-|q|})^{n-1}$

Finally, we proved :

$$\begin{aligned} \|\sum_{\underline{i}} p_{\underline{i}}(Y_1, \dots, Y_N) \otimes p_{\underline{i}}^*(Y_1, \dots, Y_N)\|_R &\leq \left((1 - |q|) \prod_{k=1}^{\infty} \frac{1 + |q|^k}{1 - |q|^k} \right)^n \left(R + \frac{2}{1 - |q|} \right)^{2n} N^n \\ &\leq \left(\frac{(1 - |q|)^2}{1 - 2|q|} \right)^n \left(\frac{2(2 + \epsilon)}{1 - |q|} \right)^{2n} N^n \end{aligned}$$

The last rough estimate is as above (in this proof for the estimate on $f(qN)$ and detailed in lemma 8 in [46]) and it concludes. Likewise

$$\|\sum_{\underline{i}} \partial_j p_{\underline{i}}(Y_1, \dots, Y_N) \otimes p_{\underline{i}}^*(Y_1, \dots, Y_N)\|_\rho \leq n \left(\frac{(1 - |q|)^2}{1 - 2|q|} \right)^n \left(\frac{2(2 + \epsilon)}{1 - |q|} \right)^{2n} N^n$$

Note that positivity comes from the identification of $\sum_n q^n P_n = \Gamma_q(qId)$ with the second quantization. \square

As a consequence, for q such that I_q holds (e.g. $\rho(q, N) < 1$), if ∂_j is the j -th free difference quotient with respect to X_1, \dots, X_N we have $\partial_j = \partial_j^{(q)} \# \Xi_q^{-1}$ since $\partial_j(X_i) = 1_{i=j} \Xi_q \# \Xi_q^{-1} = \delta_{i=j} 1 \otimes 1$. (recall $\#$ in this context is multiplication in $M \overline{\otimes} M^{op}$) We can also define an approximation of $\partial^{(q)} : \partial_j^{(q, Q)}(X_i) = 1_{i=j} \Xi_q^Q$.

Finally we want to introduce a derivation giving the number operator as generator of the corresponding Dirichlet form. We define first the $\hat{\partial}_j^{(q)} := \partial_k \# X_q^{k'}$ valued in $\Gamma_q(\mathcal{H} \oplus \mathcal{H})$ where $X_q^{k'}$ is the q -Gaussian variable corresponding to the second copy of the eigenvector h_k in the second term of the direct sum. Said otherwise this is the only derivation sending X_q^k to $X_q^{k'}$. This derivation is defined for any q . We also want to compare this derivation to another derivation valued in the coarse correspondence. For q such that $q\sqrt{N} < 1$, we can define $\tilde{\partial}_j^{(q)} := \partial_j \# \Xi_q^{1/2}$.

Proposition 43. *For any $\xi \in \mathcal{H}^{\otimes n}, \eta \in \mathcal{H}^{\otimes m}$, any $q \in (-1, 1)$, $\psi(\xi) \in D(\hat{\partial}_k^{(q)})$ and we have :*

$$\sum_k \langle \hat{\partial}_k^{(q)}(\xi), \hat{\partial}_k^{(q)}(\eta) \rangle_q = n \langle \xi, \eta \rangle_q.$$

As a consequence, $\hat{\partial}^{(q)} = (\hat{\partial}_1^{(q)}, \dots, \hat{\partial}_n^{(q)})$ is a closable derivation, with $\hat{\partial}^{(q)*} \hat{\partial}^{(q)} = \tilde{\Delta}$ the number operator satisfying $\tilde{\Delta}(\xi) = n\xi, \xi \in \mathcal{H}^{\otimes n}$.

Moreover if $q\sqrt{N} < 1$, for any polynomial $P, Q, R, S \in \mathbb{C}\langle X_1, \dots, X_n \rangle$,

$$\langle R\hat{\partial}_k^{(q)}(P), S\hat{\partial}_k^{(q)}(Q) \rangle = \langle R\tilde{\partial}_k^{(q)}(P), S\tilde{\partial}_k^{(q)}(Q) \rangle.$$

Thus, one can see $\hat{\partial}_k^{(q)}$ as valued in a bimodule included in the coarse correspondence.

Finally, if I_q holds, $\tilde{\partial}_k^{(q)}, \partial_k^{(q)}, \partial_j^{(q,Q)}$ (for $Q \geq Q_0$ large enough depending on q such that the left hand side of the third inequality below is positive) and ∂_k are all closable and their closures share the same domain, with, for any x in their common domain :

$$\|\partial_k^{(q)}(x)\|_2 \leq \|\Xi_q^{1/2}\|_{M \otimes M^{op}} \|\tilde{\partial}_k^{(q)}(x)\|_2 \leq \|\Xi_q^{1/2}\|_{M \otimes M^{op}}^2 \|\partial_k(x)\|_2$$

$$\|\partial_k(x)\|_2 \leq \|\Xi_q^{-1/2}\|_{M \otimes M^{op}} \|\tilde{\partial}_k^{(q)}(x)\|_2 \leq \|\Xi_q^{-1/2}\|_{M \otimes M^{op}}^2 \|\partial_k^{(q)}(x)\|_2.$$

$$\|\partial_k^{(q)}(x)\|_2 (1 - \|\Xi_q^Q - \Xi_q\|_{M \otimes M^{op}} \|\Xi_q^{-1/2}\|_{M \otimes M^{op}}^2) \leq \|\partial_k^{(q,Q)}(x)\|_2 \leq \|\Xi_q^Q\|_{M \otimes M^{op}} \|\tilde{\partial}_k(x)\|_2.$$

Actually, $\partial^{(q)}, \tilde{\partial}^{(q)}, \partial, \partial^{(q,Q)}$ are equivalent (in the sense of subsection 2.1).

Proof. The domain property stated is obvious since $\psi(\xi)$ is a non-commutative polynomial in X_1, \dots, X_N . Moreover, by linearity, we need to check the first equality only for $\psi(\xi) = \psi_{j_1, \dots, j_n}$ and $\psi(\eta) = \psi_{l_1, \dots, l_p}$.

As in the proof of proposition 2.7 in [6], we use the following identity :

$$\psi_{i_1, \dots, i_n} = X_{i_1} \psi_{i_2, \dots, i_n} - \sum_{j \geq 2} q^{j-2} \delta_{i_1=i_j} \psi_{i_2, \dots, \hat{i}_j, \dots, i_n}.$$

Applying ∂_k , we find :

$$\partial_k(\psi_{i_1, \dots, i_n}) = 1_{i_1=k} \otimes \psi_{i_2, \dots, i_n} + X_{i_1} \partial_k(\psi_{i_2, \dots, i_n}) - \sum_{j \geq 2} q^{j-2} \delta_{i_1=i_j} \partial_k(\psi_{i_2, \dots, \hat{i}_j, \dots, i_n}).$$

As a consequence we deduce by an immediate induction : $\partial_k(\psi_{i_1, \dots, i_n}) \# X_q^{k'} = \sum_j 1_{i_j=k} \psi_{i_1, \dots, \hat{i}_j, \dots, i_n}$ (where the prime indicates we have to consider the i_j of the second copy of \mathcal{H}).

We can thus compute (using the definition of the scalar product in the second and fourth lines, and removing properly summations and Kronecker functions $1_{a=b}$ in the third and fifth lines):

$$\begin{aligned} \sum_k \langle \hat{\partial}_k^{(q)}(\psi_{j_1, \dots, j_n}), \hat{\partial}_k^{(q)}(\psi_{l_1, \dots, l_m}) \rangle_q &= \sum_k \sum_{i, \ell} 1_{j_i=k=\ell} \langle \psi_{j_1, \dots, \hat{j}_i, \dots, j_n}, \psi_{l_1, \dots, \hat{l}_\ell, \dots, l_m} \rangle_q \\ &= 1_{\{n=m\}} \sum_k \sum_{i, \ell} 1_{j_i=k=\ell} \sum_{\pi \in S_n} q^{i(\pi)} 1_{\sigma(i)=\ell} \prod_{p=1}^n 1_{j_p=l_{\sigma(p)}} \\ &= 1_{\{n=m\}} \sum_k \sum_i 1_{j_i=k} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{p=1}^n 1_{j_p=l_{\sigma(p)}} \\ &= 1_{\{n=m\}} \sum_k \sum_i 1_{j_i=k} \langle \psi_{j_1, \dots, j_n}, \psi_{l_1, \dots, l_m} \rangle_q \\ &= n \langle \psi_{j_1, \dots, j_n}, \psi_{l_1, \dots, l_m} \rangle_q. \end{aligned}$$

We now assume $q\sqrt{N} < 1$. To explain the second equality, note that we can rewrite (since Ξ_q selfadjoint) $\langle a \otimes b \# \Xi_q^{1/2}, a' \otimes b' \# \Xi_q^{1/2} \rangle = \langle a \otimes b, a' \otimes b' \# \Xi_q \rangle$ and then :

$$\langle a \otimes b, a' \otimes b' \# \Xi_q \rangle = \sum_n q^n \sum_\xi \tau(a^* a' \xi) \tau(\xi^* b' b^*) = \sum_n q^n \tau(a^* a' P_n(b' b^*)) = \tau(a^* a' \Gamma_q(qId)(b' b^*)),$$

where $\Gamma_q(qId)$ is the second quantization. Then, our claim follows for instance from Theorem 3.2 in [15] which implies $\tau(a^* a' \Gamma_q(qId)(b' b^*)) = \langle a \otimes b \# X'_k, a' \otimes b' \# X'_k \rangle$ (Theorem 3.2 is a variant of Ito formula, one can apply it after identifying the first copy of \mathcal{H} with $\text{span}\{\sqrt{n}1_{[k/2n, (k+1)/2n)}, k = 1, \dots, n\}$ in $L^2([0, 1])$ and the second with $\text{span}\{\sqrt{n}1_{[k/2n, (k+1)/2n)}, k = n+1, \dots, 2n\}$).

The last inequalities in the proposition on non-commutative polynomials follow from corollary 42 and assumption I_q . It implies closability since $\partial^{(q)}$ is closable (by lemmas 12 and 40) and the result extended to the closures. \square

Remark 44. Even if we won't use later the analytic bound we got in Corollary 42, it is worth noting it can enable us using our last derivation $\tilde{\partial}^{(q)}$ to prove complete metric approximation property for $\Gamma_q(\mathcal{H})$ with small q , or (reprove) absence of non-trivial projections for the corresponding C^* -algebras following the lines of [20]. Indeed, first note that using the analytic expansion $\Xi_q^{1/2}(Y_1, \dots, Y_N) = 1 \otimes 1 + \sum_{k=1}^\infty \binom{1/2}{k} (\Xi_q(Y_1, \dots, Y_N) - 1 \otimes 1)^k$ so that we get a Lipschitz bound

$$\begin{aligned} & \|\Xi_q^{1/2}(Y_1, \dots, Y_N) - \Xi_q^{1/2}(Z_1, \dots, Z_N)\| \\ & \leq \sum_{k=1}^\infty \left| \binom{1/2}{k} \right| k \|\Xi_q - 1 \otimes 1\|_\rho^{k-1} \sum_i (\|\partial_i \otimes 1(\Xi_q)\|_\rho + \|1 \otimes \partial_i(\Xi_q)\|_\rho) \|X_i - Z_i\| \\ & \leq \frac{1}{2\sqrt{1 - \|(\Xi_q - 1 \otimes 1)\|_\rho}} \sup_i (\|X_i - Z_i\|) 2 \frac{4(2+\epsilon)^2 N |q| (1-2|q|)}{(1 - (2+4(2+\epsilon)^2 N)|q|)^2} \\ & \leq \frac{1}{\sqrt{1 - (2+8(2+\epsilon)^2 N)|q|}} \sup_i (\|X_i - Z_i\|) \frac{4(2+\epsilon)^2 N |q| (1-2|q|)}{(1 - (2+4(2+\epsilon)^2 N)|q|)^{3/2}} \\ & \leq \kappa \sup_i (\|X_i - Z_i\|), \end{aligned}$$

the last inequality being true for $\kappa < 1/2$ if $4(2+\epsilon)^2 N |q| (1-2|q|) \leq 4(2+\epsilon)^2 N |q| < (1 - 2(2+8(2+\epsilon)^2 N)|q|)/2 \leq (1 - (2+8(2+\epsilon)^2 N)|q|)^2/2$, i.e. for $(4+24(2+\epsilon)^2 N)|q| < 1$.

As in [20], one can consider the solutions (given by Picard iteration) $X_{i,t} = X_i - \frac{1}{2} \int_0^t ds X_{i,s} + \int_0^s \Xi_q^{1/2}(X_{1,s}, \dots, X_{N,s}) \# dS_s^i$, $Y_{i,t} = 0 - \frac{1}{2} \int_0^t ds Y_{i,s} + \int_0^s \Xi_q^{1/2}(Y_{1,s}, \dots, Y_{N,s}) \# dS_s^i$. From [46] or [12] (and the above proposition 43), $X_{i,t}$ is stationary so that $\alpha_t(X_i) = X_{i,t}$ ($i = 1, \dots, N$) defines a trace preserving homomorphism. By variation of constants, one gets

$$\begin{aligned} (X_{i,t} - Y_{i,t}) &= X_i - \frac{1}{2} \int_0^t ds (X_{i,s} - Y_{i,s}) + \int_0^s (\Xi_q^{1/2}(X_{1,s}, \dots, X_{N,s}) - \Xi_q^{1/2}(Y_{1,s}, \dots, Y_{N,s})) \# dS_s^i \\ &= e^{-1/2t} X_i + \int_0^t e^{-1/2(t-s)} (\Xi_q^{1/2}(X_{1,s}, \dots, X_{N,s}) - \Xi_q^{1/2}(Y_{1,s}, \dots, Y_{N,s})) \# dS_s^i \end{aligned}$$

And from our inequality above and Biane-Speicher's L^∞ version of Burkholder-Gundy inequality we deduce :

$$\sup_i \|X_{i,t} - Y_{i,t}\| \leq e^{-t/2} \sup_i \|X_i\| + \left(\int_0^t ds e^{-(t-s)} \kappa^2 \sup_i \|X_{i,s} - Y_{i,s}\|^2 \right)^{1/2}$$

so that from Gronwall's lemma (in line 2 after using a trivial bound on squares and $\kappa < 1/2$) :

$$\begin{aligned} \sup_i \|X_{i,t} - Y_{i,t}\|^2 &\leq 2e^{-t} \sup_i \|X_i\|^2 + 1/2 \left(\int_0^t ds e^{-(t-s)} \sup_i \|(X_{i,s} - Y_{i,s})\|^2 \right) \\ &\leq 2e^{-t/2} \sup_i \|X_i\|^2 \rightarrow 0 \end{aligned}$$

Thus, since $Y_{i,s} \in C^*(S_t^i)$ we got the property of corollary 4.1 in [20] and by the reasoning of Theorem 4.2 there, $C^*(X_1, \dots, X_N)$ has no non-trivial projections (remember this applies when $(4 + 24(2 + \epsilon)^2 N)|q| < 1$). Likewise by the reasoning of theorem 4.3 in [20] we get complete metric approximation property in the way they get Haagerup property. This result has been recently extended by Stephen Avsec to all $q \in (-1, 1)$. Of course, in the smaller range of q we consider we have almost inclusion in $L(\mathbb{F}_\infty)$ too.

4.2.3. Regularity for Ξ_q and almost coassociativity of two among our three derivations. We start by noting the following consequence of proposition 43 (we uses a notation from the beginning of subsection 2.1.5) :

Lemma 45. *If I_q holds and for $\xi \in \mathcal{H}^{\otimes n}$, for \mathcal{D} any among $\overline{\partial_{m_p}^{(k_p, p-k_p)}} \circ \dots \circ \overline{\partial_{m_1}^{(k_1, 1-k_1)}}$ $p \in [1, n], k_l \in [1, p], m_l \in [1, N], l = 1 \dots p$, $\|\mathcal{D}(\xi)\|_2^2 \leq (n \|\Xi_q^{-1}\|_{M \overline{\otimes} M^{op}})^p \|\xi\|_2^2$ ■*

Lemma 46. *Assume I_q and $|q|N < 1$, then $\|\partial_k \otimes 1 \Xi_q\|_{(M \overline{\otimes} M^{op}) \hat{\otimes} M} < \infty$. Likewise, with U, V, W any among $\partial_j, \partial_i^{(q)}$, we have $U \otimes V(\Xi_q) \in (M \overline{\otimes} M^{op}) \hat{\otimes} (M^{op} \overline{\otimes} M)$, $(U \otimes 1 \otimes 1)(V \otimes 1)(\Xi_q)$, $(1 \otimes U \otimes 1)(V \otimes 1)(\Xi_q) \in (M \overline{\otimes} M^{op} \overline{\otimes} M) \hat{\otimes} M$, $(U \otimes 1 \otimes W)(V \otimes 1)(\Xi_q)$, $(1 \otimes U \otimes W)(V \otimes 1)(\Xi_q) \in (M \overline{\otimes} M^{op} \overline{\otimes} M) \hat{\otimes} (M \overline{\otimes} M^{op})$ and symmetric variants. We also have convergence of any $\Xi_q^{(Q)}$ variant to the Ξ_q case in those spaces.*

Assume now only I_q , then, for \mathcal{D} any among $\overline{\partial_{m_p}^{(k_p, 1+p-k_p)}} \circ \dots \circ \overline{\partial_{m_1}^{(k_1, 2-k_1)}}$ $p \geq 1, k_l \in [1, p+1], m_l \in [1, N], l = 1 \dots p$, $\Xi_q, \Xi_q^Q \in D(\mathcal{D})$ and $\|\mathcal{D}(\Xi_q)\|_{M \overline{\otimes} p \overline{\otimes} M^{op}}, \|\mathcal{D}(\Xi_q^Q)\|_{M \overline{\otimes} p \overline{\otimes} M^{op}} \leq (2C_{|q|})^{3(p+1)/2} \|\Xi_q^{-1}\|_{M \overline{\otimes} M^{op}}^p g(q\sqrt{N}, p)$ (for any $p > 0$, $g(x, p)$ an analytic function of radius of convergence 1 in x), $\|\mathcal{D}(\Xi_q^{Q'} - \Xi_q)\|_{M \overline{\otimes} p \overline{\otimes} M^{op}} \rightarrow_{Q' \rightarrow \infty} 0$

Proof. We compute (the first inequality bellow is obvious from lemma 42, the second equality comes from proposition 43) :

$$\|\partial_k(p_i)\|_{L^2(M, \tau_q) \otimes L^2(M, \tau_q)}^2 \leq \|\Xi_q^{-1/2}\|^2 \langle \partial_k(p_i) \# \Xi_q, \partial_k(p_i) \rangle = \|\Xi_q^{-1/2}\|^2 |i|$$

Now one can use (14) and (15) in the second line and this in the third to conclude to the first result.

$$\begin{aligned}
\|\partial_k \otimes 1\Xi_q\|_{(M\overline{\otimes}M^{op})\hat{\otimes}M} &\leq \sum_n q^n \sum_{|\underline{i}|=n} \|\partial_k(p_{\underline{i}})\|_{M\overline{\otimes}M^{op}} \|p_{\underline{i}}\|_M \\
&\leq C_{|q|}^{9/2} \sum_n |q|^n \sum_{|\underline{i}|=n} (n+1)^4 \|p_{\underline{i}}\|_2 \|\partial_k(p_{\underline{i}})\|_{L^2(M\overline{\otimes}M)} \\
&\leq C_{|q|}^{9/2} \sum_n |q|^n (n+1)^4 \|\Xi_q^{-1/2}\| n^{1/2} N^n
\end{aligned}$$

As stated, as soon as $\|\Xi_q^{-1/2}\| < \infty$ and $|q|N < 1$, this sum is finite. The proof of $\partial_j \otimes \partial_i^{(q)}(\Xi_q) \in (M\overline{\otimes}M^{op})\hat{\otimes}(M^{op}\overline{\otimes}M)$ is really similar. For the last statement, we need likewise a bound e.g. on $\|\partial_j \otimes 1\partial_i^{(q)}(p_{\underline{j}})\|_{M\overline{\otimes}M^{op}\overline{\otimes}M} \leq \|\partial_j \otimes 1(\Xi_q)\|_{(M\overline{\otimes}M^{op})\hat{\otimes}M} \|\partial_i(p_{\underline{j}})\|_{M\overline{\otimes}M} + \|\Xi_q\|_{M^{op}\overline{\otimes}M} \|\partial_j \otimes 1\partial_i(p_{\underline{j}})\|_{M\overline{\otimes}M^{op}\overline{\otimes}M}$ coming from the previous lemma (with a 3 tensor product variant of Bożejko inequality and lemma 43).

For the second statement we bound using Bożejko inequality and the previous lemma :

$$\begin{aligned}
\|\mathcal{D}(\Xi_q^Q)\|_{M\overline{\otimes}P\overline{\otimes}M^{op}} &\leq \sum_{n=0}^Q q^n \|\mathcal{D}(\sum_{|\underline{i}|=n} p_{\underline{i}} \otimes p_{\underline{i}}^*)\|_{M\overline{\otimes}P\overline{\otimes}M^{op}} \\
&\leq \sum_{n=0}^Q q^n (2n+1)^{3(p+1)/2} C_{|q|}^{3(p+1)/2} (n \|\Xi_q^{-1}\|_{M\overline{\otimes}M^{op}})^p \|\sum_{|\underline{i}|=n} p_{\underline{i}} \otimes p_{\underline{i}}^*\|_2 \\
&\leq (2C_{|q|})^{3(p+1)/2} \|\Xi_q^{-1}\|_{M\overline{\otimes}M^{op}}^p \sum_{n=0}^{\infty} (q\sqrt{N})^n (n+1)^{3p+2}
\end{aligned}$$

The remaining statements follow similarly. \square

We now prove almost coassociativity.

Lemma 47. *If I_q holds, $\partial^{(q)}$ and $\partial^{(q,Q)}$ (resp. ∂ and $\partial^{(q)}$) are almost coassociative with respect to ∂ (resp. $\partial \oplus \partial^{(q)}$) with defect $C = (C_{ij}^k = 1_{j=k}\partial_i \otimes 1\Xi_q, C_{ji}^k = -1_{j=k}1 \otimes \partial_i\Xi_q)$ and $C^{(Q)} = (C_{ij}^{(Q)k} = 1_{j=k}\partial_i \otimes 1\Xi_q^{(Q)}, C_{ji}^{(Q)k} = -1_{j=k}1 \otimes \partial_i\Xi_q^{(Q)})$, (resp. $D = (D_{ij}^k = -1_{k+N=i>N}1 \otimes \partial_j\Xi_q, D_{ji}^k = 1_{k+N=i>N}\partial_j \otimes 1\Xi_q)_{i,k=1..2N, j=1..N}$ and $C' = (C_{ij}^{k'} = 1_{k=j, i \leq N}\partial_i \otimes 1\Xi_q + 1_{k=j, i > N}\partial_i^{(q)} \otimes 1\Xi_q - 1_{k+N=i>N}1 \otimes \partial_j^{(q)}\Xi_q, C_{ji}^{k'} = -1_{k=j, i \leq N}1 \otimes \partial_i\Xi_q - 1_{i>N}D_{ij}^k)$). If I_q holds and $|q|N < 1$, $\partial^{(q)}$ (as giving δ), $\partial^{(q,Q)}$ (as giving $\delta_{(Q)}$) ∂ (as giving $\tilde{\delta}$) satisfy assumption 2 (a) (Q_0 taken as the starting value of their equivalence as derivations given in lemma 43).*

More generally, for any $U \in (\mathcal{C}\langle X_1, \dots, X_N \rangle) \otimes_{alg} \mathcal{C}\langle X_1, \dots, X_N \rangle^{opN}$, again if I_q holds and $|q|N < 1$, then $\partial^{(q,U,\epsilon)} = \epsilon\partial^{(q)} \oplus \partial^{(q)} \# U$ (giving δ), $\partial^{(q,Q,U,\epsilon)} = \epsilon\partial^{(q,Q)} \oplus \partial^{(q,Q)} \# U$ (as giving $\delta_{(Q)}$) and $\partial \oplus \partial$ (giving $\tilde{\delta}$) satisfy assumption 2 (a) (same Q_0).

Proof. Recall $D(\partial^{(q)}) = D(\partial)$ is nothing but non-commutative polynomials. It suffices to check the relations on generators (since both sides are derivations). Assumption 2 (a) then comes from lemma 46. The case with U is easily deduced. Note that the boundedness of $(1 \otimes \tau)\partial_j^{(q)}$ in assumption 2 follows from lemma 40, the variant with U is also easy from this. \square

Lemma 48. Assume I_q , then with the notation of the previous lemma $D_{i,j}^k \in D(1 \otimes \partial_j^{(q)*}), D_{j,i}^k \in D(\partial_j^{(q)*} \otimes 1)$. and for $j \leq N, i, k > N$, and $\partial_i^{(q)} \partial_j^{(q)*} 1 \otimes 1, 1 \otimes \partial_j^{(q)*} D_{i,j}^k = -1_{k+N=i>N} 1 \otimes \partial_j^{(q)*} \partial_j \Xi_q, \partial_j^{(q)*} \otimes 1 D_{j,i}^k = 1_{k+N=i>N} \partial_j^{(q)*} \partial_j \otimes 1 \Xi_q \in M \overline{\otimes} M^{op}$.

Proof. Note that $\partial_i^{(q)} \partial_j^{(q)*} 1 \otimes 1 = 1_{i=j} \Xi_q$ and the corresponding statement is obvious here. Since $\partial_j^{(q)*} \partial_j$ is the number operator with respect to the j -th variable (as in proposition 43) we have, using Bożejko inequality, e.g.:

$$\|\partial_j^{(q)*} \partial_j \otimes 1 \Xi_q\| \leq C_{|q|}^3 \sum_n q^n (n+1)^2 \|\sum \partial_j^{(q)*} \partial_j(p_i) \otimes p_i^*\|_2 \leq C_{|q|}^3 \sum_n q^n (n+1)^2 n N^{n/2}.$$

□

Recall in assumption 2 (b) we called $\Pi_{\pi,p}^q = (p, q, \pi, \pi) \Pi_p^q := \Pi_{p,p}^q$ and we checked in remark 21 that $Comp_{\Pi_\infty^2}(R_{\infty,\epsilon}, 2, 2, 2)$ holds with $R_{\infty,\epsilon} = (2 + \epsilon, 2 + \epsilon, 1 + \epsilon/2, 1 + \epsilon/2, 1 + \epsilon/2)$. with $\tilde{\rho} = 2$.

Lemma 49. Assume I_q holds.

$\mathcal{C}\langle X_1, \dots, X_N \rangle \subset [\cap_{Q \in [Q_0, \infty)} (B_{\Pi_\infty^2, R_\infty^\epsilon}^3(M_0, \partial, \partial^{(q)}, \partial^{(q,Q)}) \cap B_{\infty, \infty, 2, 2}^{2, sym}(M_0, \partial, \partial^{(q,Q)})) \cap B_{\Pi_\infty^2, R_\infty^\epsilon}^3(M_0, \partial \oplus \partial^{(q)}, \partial^{(q)}, \partial^{(q)}, \partial) \cap B_{\infty, \infty, 2, 2}^{2, sym}(M_0, \partial, \partial^{(q)}) \cap B_{\infty, \infty, 2, 2}^{2, sym}(M_0, \partial \oplus \partial^{(q)}, \partial^{(q)})]$.

As a consequence, ∂ (as $\tilde{\partial}$) and $\partial^{(q)}, \partial^{(q,Q)}$ (as $\partial, \partial_{(Q)}$) satisfy assumption 2 (b), (c) (in the context $Comp_{\Pi_\infty^2}(R_{\infty,\epsilon}, 2, 2, 2)$, same Q_0 as in lemma 47). Moreover, $\partial \oplus \partial_{(q)}$ satisfy $0_{\infty, \infty}^{a \partial^{(q)} 2}$.

More generally, for any $U \in (\mathcal{C}\langle X_1, \dots, X_n \rangle)^{\otimes_{alg} 2)^N}$, then $\partial^{(q,U,\epsilon)} = \epsilon \partial^{(q)} \oplus \partial^{(q)} \# U$ (giving δ), $\partial^{(q,Q,U,\epsilon)} = \epsilon \partial^{(q,Q)} \oplus \partial^{(q,Q)} \# U$ (as giving $\delta_{(Q)}$) and $\partial \oplus \partial$ (giving $\tilde{\delta}$) satisfy assumption 2 (b), (c).

Proof. The right hand side of the inclusion is an algebra by lemma 22. The fact that X_i 's belong to this intersection follows from lemma 46. Assumption 2 (c) follows with $R = R_\infty^\epsilon$. For assumption 2 (b) the only properties it remains to check are those about reducibility, equivalence, with $2\pi_{(2)}/(\pi_{(2)} - 2) = 2$ ($\pi_{(2)} = \infty$). The coefficients of reducibility of ∂ to $\partial^{(q)}$ are $i_j = j, I_j = \Xi_q^{-1}, k_j = j, K_j = \Xi_q$, and $K_j^Q = \Xi_q^Q$ (for the coefficients of equivalence between ∂ and $\partial^{(q,Q)}$). All the supplementary conditions follow from lemma 46. For the supplementary statement, we use the fact $\partial \oplus \partial_{(q)}$ is reducible to $\partial_{(q)} \oplus \partial_{(q)}$ (which has the required $\partial_{(q)}^* 1 \otimes 1 \in M$) 2-regularly with respect to $\partial_{(q)}$.

In the context with U , B^3 spaces don't change thanks to the component with ϵ , (c) is thus deduced as before by inclusion of non-commutative polynomials. We have to make explicit equivalences and reducibility in (b). The coefficients of reducibility of $\partial \oplus \partial$ to $\partial^{(q,U,\epsilon)}$ are $i_j = j$ if $j \leq N, i_j = j - N$ otherwise; $I_j = 1/\epsilon \Xi_q^{-1}; k_j = j; K_j = \epsilon \Xi_q$ if $j \leq N, K_j = \Xi_q \# U$ otherwise. For the coefficients of equivalence between $\partial \oplus \partial$ and $\partial^{(q,Q,U,\epsilon)}$ we have analogously $i_j = j$ if $j \leq N, i_j = j - N$ otherwise; $I_j^{(Q)} = 1/\epsilon \Xi_q^{-1} (1 \otimes 1 - (\Xi_q - \Xi_q^Q) \Xi_q^{-1})^{-1}$ (recall Q_0 chosen to make this invertible); $k_j = j; K_j^{(Q)} = \epsilon \Xi_q^Q$ if $j \leq N, K_j = \Xi_q^Q \# U$ otherwise. $\partial_j^{(q,U,\epsilon)*} 1 \otimes 1$ exist and are in M using lemma 12. □

4.3. An example of Lipschitz conjugate variable: q -Gaussian families for small q . We now want to play with the three previous derivations to get regularity results for conjugate variables.

Theorem 50. *Assume $\rho(q, N) < 1$ as defined in corollary 42 (e.g. $|q|\sqrt{N} \leq 0.13$) and $|q|N < 1$ then q -Gaussian variables have finite free Fisher information $\Phi^*(X_1, \dots, X_N) < \infty$ (and actually the conjugate variable is in the domain of the L^2 -closure of the free difference quotient).*

Furthermore assume also condition $\nu(q, N) < 1$ in corollary 42 (e.g. $|q|N \leq 0.13$), in that case the conjugate variables are in $\Gamma_q(\mathcal{H})$ and X_1, \dots, X_N have even Lipschitz conjugate variables. As a consequence, under condition $\nu(q, N) < 1$ we have $\delta_0(X_1, \dots, X_N) = N$.

Remark 51. In [46], Shlyakhtenko proved $\delta_0(X_1, \dots, X_N) \rightarrow N$ when $q \rightarrow 0$, we can prove this value is identically equal to N on a small neighborhood of 0. Actually, he proved $\delta_0(X_1, \dots, X_N) \geq N \left(1 - \frac{q^2 N}{1 - q^2 N}\right)$ for $|q| < (4N^3 + 2)^{-1}$. Here the improvement in terms of value of δ_0 mainly comes from using a better derivation in that respect (the free difference quotient). The improvement in terms of values of q comes from the fact we only need a Lipschitz condition instead of a analyticity condition on the conjugate variable. However, in considering like us the free difference quotient and with a better estimate of the domain of analyticity, one would also get a range of order $|q| < 1/CN$ in inverse of the number of generators (with a huger C than ours, cf. Rmk 44). Note finally that corollary 2.11 in [44] implies $\delta^*(X_1, \dots, X_N) = N$ as soon as I_q holds, thus e.g. assuming only $\rho(q, N) < 1$.

Proof. Let $M = \Gamma_q(\mathcal{H})$.

Step 1: Finite Fisher Information under $|q|N < 1$ and $\rho(q, N) < 1$

Recall the notation introduced before Corollary 42 so that $\iota\gamma\beta\alpha$ is the natural map from $M \hat{\otimes} M^{op}$ to $L^2(M \otimes M^{op})$ (we may use later implicitly).

Claim : $\iota\gamma\beta\alpha(M \hat{\otimes} M^{op}) \subset D(\partial_j^{(q)*})$ and for any $a, b \in M$

$$\partial_j^{(q)*}(a \otimes b) = aX_jb - r(h_j)^*(a)b - a(l(h_j)^*(b)).$$

Proof of Claim. As reminded in lemma 40, $1 \otimes \tau\partial_j^{(q)} = r(h_j)^*$. Moreover, since $\partial_j^{(q)}$ is a real derivation for any $x \in D(\partial_j^{(q)})$, we have $1 \otimes \tau\partial_j^{(q)}(x^*) = (\tau \otimes 1)(\partial_j^{(q)}(x))^*$. Thus if J denotes the antilinear isometry extending $J(x) = x^*$ to $L^2(M)$, we have $\tau \otimes 1\partial_j^{(q)} = J1 \otimes \tau\partial_j^{(q)}J = Jr(h_j)^*J = l(h_j)^*$. The last equality follows from formulas for annihilation operators and $J\psi_{i_1, \dots, i_n} = \psi_{i_n, \dots, i_1}$.

From lemma 12 and $\partial_j^{(q)*}(1 \otimes 1) = X_j$, one deduces for $a, b \in D(\partial_j^{(q)}) \cap M$, $\partial_j^{(q)*}(a \otimes b) = aX_jb - r(h_j)^*(a)b - a(l(h_j)^*(b))$, so that

$$\begin{aligned} \|\partial_j^{(q)*}(a \otimes b)\|_2 &\leq \|a\| \|b\| \|X_j\| + \|r(h_j)\|_{B(L^2(M))} \|a\|_2 \|b\| + \|l(h_j)\|_{B(L^2(M))} \|b\|_2 \|a\| \\ &\leq 4\|a\| \|b\| / \sqrt{1 - |q|}. \end{aligned}$$

Now for any $a, b \in M$, if $\eta_\alpha = \alpha(\alpha + \partial_j^{(q)*} \partial_j^{(q)})^{-1}$ the completely positive (thus contractive on M) resolvent associated to the generator of the corresponding Dirichlet form, we have for any $x \in M$, $\eta_\alpha(x) \in D(\partial_j^{(q)}) \cap M$ and $\|\eta_\alpha(x) - x\|_2 \rightarrow 0$ when $\alpha \rightarrow \infty$. Since $\|\partial_j^{(q)*}(\eta_\alpha(a) \otimes \eta_\alpha(b))\|_2 \leq 4\|a\|\|b\|/\sqrt{1-|q|}$ we have weak convergence in L^2 up to extraction and as $\eta_\alpha(a) \otimes \eta_\alpha(b) \rightarrow a \otimes b \in L^2(M \otimes M)$, we get $a \otimes b$ in the domain of the closed operator $\partial_j^{(q)*}$ with the formula and inequality above remaining true. This concludes. \square

Note that assuming $\nu(q, N) < 1$, one thus deduces $\Xi_q^{-1} \in D(\partial_j^{(q)*})$ with the formula :

$$\partial_j^{(q)*}(\Xi_q^{-1}) = \Xi_q^{-1} \# X_j - m(r(h_j)^* \otimes 1 + 1 \otimes l(h_j)^*)(\Xi_q^{-1}).$$

Since $(\Xi_q)^* = \Xi_q \in M \overline{\otimes} M^{op}$, we have thus shown our first result about finite Fisher information in this case.

First recall $\{h_i\}_{i=1}^N \subset \mathbb{R}^N$ is an orthonormal basis. We write for $\underline{i} = (i_1, \dots, i_n) \in N^n$ $\psi_{\underline{i}} = \psi(h_{i_1} \otimes \dots \otimes h_{i_n})$. We define the length $|\underline{i}| = n$.

We now want to prove finite Fisher information under the less restrictive condition $\rho(q, N) < 1$, $|q|N < 1$. We need to show $\Xi_q^{-1} \in D(\partial_i^{(q)*})$ and we only know from lemma 46 : $\Xi_q \in M \hat{\otimes} M^{op}$, $\partial_k \otimes 1 \Xi_q \in (M \overline{\otimes} M^{op}) \hat{\otimes} M$, $1 \otimes \partial_k \Xi_q \in M^{op} \hat{\otimes} (M \overline{\otimes} M^{op})$, $1 \otimes \partial_i^{(q)} \Xi_q \in M \overline{\otimes} M^{op} \overline{\otimes} M$, $\partial_i^{(q)} \otimes 1 \Xi_q \in M \overline{\otimes} M^{op} \overline{\otimes} M^{op}$, $\Xi_q^{-1} \in M \overline{\otimes} M^{op}$, $\partial_j \otimes \partial_i^{(q)}(\Xi_q) \in (M \overline{\otimes} M^{op}) \hat{\otimes} (M^{op} \overline{\otimes} M)$, $\partial_i^{(q)} \otimes \partial_j(\Xi_q) \in (M^{op} \overline{\otimes} M) \hat{\otimes} (M \overline{\otimes} M^{op})$, $\partial_j \otimes 1 \otimes 1 \partial_i^{(q)} \otimes 1(\Xi_q) \in (M \overline{\otimes} M^{op} \overline{\otimes} M^{op}) \hat{\otimes} M$, $1 \otimes (1 \otimes 1 \otimes \partial_j) \partial_i^{(q)}(\Xi_q) \in M \hat{\otimes} (M^{op} \overline{\otimes} M^{op} \overline{\otimes} M^{op})$ (the norms of those quantities bellow are always taken in those spaces if not otherwise specified)

Let us call $U_n = \sum_{i=0}^n (-1)^i (\Xi_q - 1 \otimes 1)^i$ (power in $M \hat{\otimes} M^{op}$) so that we know $U_n \rightarrow \Xi_q^{-1}$ in L^2 , $U_n \in M \hat{\otimes} M^{op}$ and by our first claim $U_n \in D(\partial_i^{(q)*})$. Since $\partial_i^{(q)*}$ is closed it suffices to show $\partial_i^{(q)*}(U_n)$ bounded in L^2 to get a weak limit up to extraction and $\Xi_q^{-1} \in D(\partial_i^{(q)*})$ and to get also $\Xi_q^{-1} \in D(\partial_j \partial_i^{(q)*})$, it suffices to bound $\partial_j \partial_i^{(q)*}(U_n)$ (since such a bound gives also a bound on $\|\partial_i^{(q)*}(U_n)\|_2^2 = \langle \partial_i^{(q)*} \partial_i^{(q)*}(U_n), U_n \rangle$, we only sketch the proof of both at once).

This is mainly a computation using U_n is almost an inverse and thus will behave almost as inverse when computing derivatives coming from application of ∂ . The second key point will be that, apart from a bunch of terms we can gather in something of the form $\partial_i^{(q)*}(U_n)$, the ∂_j will enable us to use only a bound on terms coming from U_n in von Neumann norm. Recall notation $\#_i$ was introduced before Corollary 42. We get (after using our formula for $\partial_i^{(q)*}$, we mainly use derivation property of ∂_j and changes of summation) :

$$\begin{aligned} \partial_j \partial_i^{(q)*}(U_n) &= \partial_j(U_n \# X_i - m \circ (1 \otimes \tau \otimes 1)(\partial_i^{(q)} \otimes 1(U_n) + 1 \otimes \partial_i^{(q)}(U_n))) \\ \partial_j(U_n \# X_i) &= 1_{i=j} U_n + \sum_{i=1}^n (-1)^i \sum_{k=0}^{i-1} \\ &(\Xi_q - 1 \otimes 1)^k \# (\partial_j \otimes 1(\Xi_q) \#_2 ((\Xi_q - 1 \otimes 1)^{i-k-1} \# X_i) + 1 \otimes \partial_j(\Xi_q) \#_1 ((\Xi_q - 1 \otimes 1)^{i-k-1} \# X_i)) \\ &= 1_{i=j} U_n - \sum_{k=0}^{n-1} (-1)^k (\Xi_q - 1 \otimes 1)^k \# (\partial_j \otimes 1(\Xi_q) \#_2 (U_{n-k-1} \# X_i) + 1 \otimes \partial_j(\Xi_q) \#_1 (U_{n-k-1} \# X_i)) \end{aligned}$$

$$\begin{aligned}
& \partial_j(m \circ (1 \otimes \tau \otimes 1))(\partial_i^{(q)} \otimes 1(U_n)) \\
&= -\partial_j m \circ 1 \otimes \tau \otimes 1 \left[\sum_{k=0}^{n-1} (-1)^k (\Xi_q - 1 \otimes 1)^k \# \left(\partial_i^{(q)} \otimes 1(\Xi_q) \#_2(U_{n-k-1}) \right) \right] \\
&= -\sum_{k=0}^{n-1} (-1)^k \left\{ (1 \otimes (m \circ 1 \otimes \tau \otimes 1)) \left[(\partial_j \otimes 1(\Xi_q - 1 \otimes 1))^k \#_2 \left(\partial_i^{(q)} \otimes 1(\Xi_q) \#_2(U_{n-k-1}) \right) \right] \right. \\
&\quad + ((m \circ 1 \otimes \tau \otimes 1) \otimes 1) \left[(1 \otimes \partial_j(\Xi_q - 1 \otimes 1))^k \#_1 \left(\partial_i^{(q)} \otimes 1(\Xi_q) \#_2(U_{n-k-1}) \right) \right] \\
&\quad + ((m \circ 1 \otimes \tau \otimes 1) \otimes 1)(\Xi_q - 1 \otimes 1)^k \# \left(\partial_i^{(q)} \otimes 1(\Xi_q) \#_2(1 \otimes \partial_j(U_{n-k-1})) \right) \\
&\quad + (\Xi_q - 1 \otimes 1)^k \# \left[(1 \otimes (m \circ 1 \otimes \tau \otimes 1))(\partial_j \otimes 1 \otimes 1 \partial_i^{(q)} \otimes 1(\Xi_q) \#_3 U_{n-k-1}) \right] \\
&\quad \left. + (\Xi_q - 1 \otimes 1)^k \# \left[((m \circ 1 \otimes \tau \otimes 1) \otimes 1)(\partial_i^{(q)} \otimes \partial_j(\Xi_q) \#_2 U_{n-k-1}) \right] \right\}
\end{aligned}$$

Preparing for the reintroduction of $\partial_i^{(q)*}(U_{n-k-1})$ we rewrite (a part of) the first line in our last right hand side :

$$\begin{aligned}
& \sum_{k=0}^{n-1} (-1)^k \left[(\partial_j \otimes 1(\Xi_q - 1 \otimes 1))^k \#_2 \left(\partial_i^{(q)} \otimes 1(\Xi_q) \#_2(U_{n-k-1}) \right) \right] \\
&= \sum_{k=0}^{n-1} (-1)^k \left[\sum_{l=0}^{k-1} (\Xi_q - 1 \otimes 1)^l \# (\partial_j \otimes 1(\Xi_q)) \#_2 (\Xi_q - 1 \otimes 1)^{k-l-1} \right] \#_2 \left(\partial_i^{(q)} \otimes 1(\Xi_q) \#_2(U_{n-k-1}) \right) \\
&= \sum_{l=0}^{n-2} (-1)^l \left((\Xi_q - 1 \otimes 1)^l \# (\partial_j \otimes 1(\Xi_q)) \#_2 \right. \\
&\quad \left. \left[\sum_{k=l+1}^{n-1} (-1)^{k-l} (\Xi_q - 1 \otimes 1)^{k-l-1} \# \left(\partial_i^{(q)} \otimes 1(\Xi_q) \#_2(U_{n-k-1}) \right) \right] \right) \\
&= \sum_{l=1}^{n-1} (-1)^l \left((\Xi_q - 1 \otimes 1)^l \# (\partial_j \otimes 1(\Xi_q)) \#_2 (\partial_i^{(q)} \otimes 1(U_{n-l-1})) \right)
\end{aligned}$$

(in the last line, note that the term with $l = n - 1$ is zero since $\partial_i^{(q)} \otimes 1(U_0) = 0$);

We will now write $\tilde{\tau} = m \circ 1 \otimes \tau \otimes 1$. Putting everything together and reintroducing in the last line $\partial_i^{(q)*}(U_{n-k-1})$ when useful in the right hand side :

$$\begin{aligned}
\partial_j \partial_i^{(q)*}(U_n) &= 1_{i=j} U_n + \sum_{k=0}^{n-1} (-1)^k (\Xi_q - 1 \otimes 1)^k \# (\tilde{\tau} \otimes 1) \left(\partial_i^{(q)} \otimes 1(\Xi_q) \#_2 (1 \otimes \partial_j(U_{n-k-1})) \right) \\
&+ \sum_{k=0}^{n-1} (-1)^k (\Xi_q - 1 \otimes 1)^k \# (1 \otimes \tilde{\tau}) \left(1 \otimes \partial_i^{(q)}(\Xi_q) \#_1 (\partial_j \otimes 1(U_{n-k-1})) \right) \\
&+ \sum_{k=0}^{n-1} (-1)^k (\Xi_q - 1 \otimes 1)^k \# (1 \otimes \tilde{\tau}) \left(\partial_j \otimes 1 \otimes 1 \partial_i^{(q)} \otimes 1(\Xi_q) \#_3 U_{n-k-1} + \partial_j \otimes \partial_i^{(q)}(\Xi_q) \#_2 U_{n-k-1} \right) \\
&+ \sum_{k=0}^{n-1} (-1)^k (\Xi_q - 1 \otimes 1)^k \# (\tilde{\tau} \otimes 1) \left(1 \otimes 1 \otimes \partial_j 1 \otimes \partial_i^{(q)}(\Xi_q) \#_1 U_{n-k-1} + \partial_i^{(q)} \otimes \partial_j(\Xi_q) \#_2 U_{n-k-1} \right) \\
&- \sum_{k=0}^{n-1} (-1)^k (\Xi_q - 1 \otimes 1)^k \# \left(\partial_j \otimes 1(\Xi_q) \#_2 \partial_i^{(q)*}(U_{n-k-1}) + 1 \otimes \partial_j(\Xi_q) \#_1 \partial_i^{(q)*}(U_{n-i-1}) \right)
\end{aligned}$$

We can now deduce from this a bound for $p \in [2, \infty]$ in $L^p(M \otimes M^{op})$ if we know a bound on $\|\partial_i^{(q)*}(U_k)\|_p$. Under the assumption $|q|N < 1$ we know this is finite for $p = 2$, we will use it later in the case $p = \infty$ under a stronger assumption. (the second line below corresponds to the last line of our last equation, the first and third to the first and second, the fourth and fifth to the third and fourth).

$$\begin{aligned}
\|\partial_j \partial_i^{(q)*}(U_n)\|_p &\leq 1_{i=j} \|U_n\|_p \\
&+ \left(\sup_{k \leq n-1} \|\partial_i^{(q)*}(U_k)\|_p \right) (\|\partial_j \otimes 1(\Xi_q)\| + \|1 \otimes \partial_j(\Xi_q)\|) \sum_{k=0}^{n-1} \|\Xi_q - 1 \otimes 1\|_{M \overline{\otimes} M^{op}}^k \\
&+ (\|1 \otimes \partial_i^{(q)}(\Xi_q)\| \|\partial_j \otimes 1(\Xi_q)\| + \|\partial_i^{(q)} \otimes 1(\Xi_q)\| \|1 \otimes \partial_j(\Xi_q)\|) \sum_{k=2}^n k(k-1)/2 \|\Xi_q - 1 \otimes 1\|_{M \overline{\otimes} M^{op}}^{k-2} \\
&+ (\|\partial_j \otimes \partial_i^{(q)}(\Xi_q)\| + \|\partial_j \otimes 1 \otimes 1 \partial_i^{(q)} \otimes 1(\Xi_q)\|) \sum_{k=0}^{n-1} (k+1) \|\Xi_q - 1 \otimes 1\|_{M \overline{\otimes} M^{op}}^k \\
&+ (\|\partial_i^{(q)} \otimes \partial_j(\Xi_q)\| + \|1 \otimes 1 \otimes \partial_j 1 \otimes \partial_i^{(q)}(\Xi_q)\|) \sum_{k=0}^{n-1} (k+1) \|\Xi_q - 1 \otimes 1\|_{M \overline{\otimes} M^{op}}^k
\end{aligned}$$

Since $\|\Xi_q - 1 \otimes 1\|_{M \overline{\otimes} M^{op}} < 1$ all the sums of the right hand side extended to infinity converge so that we get constants C, D $\|\partial_j \partial_i^{(q)*}(U_n)\|_2 \leq C + D \left(\sup_{k \leq n-1} \|\partial_i^{(q)*}(U_k)\|_2 \right)$ and thus $\|\partial_i^{(q)*}(U_n)\|_2^2 \leq \|\Xi_q\| \|U_n\|_2 (C + D \left(\sup_{k \leq n-1} \|\partial_i^{(q)*}(U_k)\|_2 \right))$, and a standard bound concludes to finiteness of $\sup_k \|\partial_i^{(q)*}(U_k)\|_2$.

Step 2: Bounded conjugate variable under $\nu(q, N) < 1$.

From the previous step, we know :

$$\begin{aligned} \partial_i^{(q)*}(U_n) &= U_n \# X_i + \sum_{k=0}^{n-1} (-1)^k (\Xi_q - 1 \otimes 1)^k \# \\ &\quad m \circ (1 \otimes \tau \otimes 1) \left(\partial_i^{(q)} \otimes 1(\Xi_q) \#_2(U_{n-k-1}) + 1 \otimes \partial_i^{(q)}(\Xi_q) \#_1(U_{n-k-1}) \right) \end{aligned}$$

And thus,

$$\begin{aligned} \|\partial_i^{(q)*}(U_n)\| &\leq \|U_n\|_{M \hat{\otimes} M^{op}} \|X_i\| \\ &+ \left(\|\partial_i^{(q)} \otimes 1(\Xi_q)\|_{(M \overline{\otimes} M^{op}) \hat{\otimes} M^{op}} + \|1 \otimes \partial_i^{(q)}(\Xi_q)\|_{M \hat{\otimes} (M \overline{\otimes} M^{op})} \right) \sum_{k=0}^{n-1} (k+1) \|\Xi_q - 1 \otimes 1\|_{M \hat{\otimes} M^{op}}^k \\ &\leq \|X_i\| \sum_{k=0}^{\infty} \|\Xi_q - 1 \otimes 1\|_{M \hat{\otimes} M^{op}}^k \\ &+ \left(\|\partial_i^{(q)} \otimes 1(\Xi_q)\|_{(M \overline{\otimes} M^{op}) \hat{\otimes} M^{op}} + \|1 \otimes \partial_i^{(q)}(\Xi_q)\|_{M \hat{\otimes} (M \overline{\otimes} M^{op})} \right) \sum_{k=0}^{\infty} (k+1) \|\Xi_q - 1 \otimes 1\|_{M \hat{\otimes} M^{op}}^k \end{aligned}$$

The last inequality gives a finite bound for $\nu(q, N) < 1$ as, then, by corollary 42, we have $\|\Xi_q - 1 \otimes 1\|_{M \hat{\otimes} M^{op}} < 1$. Since we showed in step 1 $\partial_i^{(q)*}(U_n) \rightarrow \partial_i^{(q)*}(\Xi_q^{-1})$ weakly in L^2 up to extraction, this means we have ultraweak convergence of the same extraction. Thus especially $\partial_i^{(q)*}(\Xi_q^{-1}) \in M$.

Step 3: Lipschitz conjugate variable under $\nu(q, N) < 1$.

Since we now know $\sup_k \|\partial_i^{(q)*}(U_k)\|_M < \infty$ from the second step, the end of the first step gives : $\|\partial_j \partial_i^{(q)*}(U_n)\|_{M \overline{\otimes} M^{op}} \leq C + D \left(\sup_k \|\partial_i^{(q)*}(U_k)\|_M \right)$.

Again since we saw in step one : $\partial_j \partial_i^{(q)*}(U_n) \rightarrow \partial_j \partial_i^{(q)*}(\Xi_q^{-1})$ weakly in L^2 up to extraction, we got $\partial_j \partial_i^{(q)*}(\Xi_q^{-1}) \in M \overline{\otimes} M^{op}$.

Putting everything together, this concludes the proof of the second part of our theorem (the statement on microstate free entropy dimension uses the R^ω embeddability result of [47] and corollary 38).

□

4.4. A better estimate on microstate free entropy dimension for q-Gaussian variables using a non-coassociative derivation. This subsection motivates the more general theory for almost coassociative derivations by an application to q-Gaussian variables.

Beyond the interest of knowing those variables do have Lipschitz conjugate variable for small q , we can get the computation of δ_0 in a slightly larger range in using directly $\partial^{(q)}$ (giving a derivation δ after extension on Brownian motion, with ∂ giving $\tilde{\delta}$, we want to check assumption 2 for them). We keep the notation of the previous subsections and only sketch the proofs.

We first want to prove a preliminary result using results of subsection 2.1 that will enable us to find a good core for Δ .

We will assume throughout $\rho < 1$ and $|q|N < 1$ so that by theorem 50, we have a conjugate variables $\tilde{\delta}_i^* 1 \otimes 1 \in L^2(M, \tau)$ and $\tilde{\delta}_i^* 1 \otimes 1 \in D(\partial)$

Recall the notation introduced before proposition 43, we consider $\tilde{\partial}_k^{(q)} = \partial_k \# \Xi_q^{1/2}$ equivalent to ∂ in the sense of g) in Γ_1 as soon as $\rho(q, N) < 1$ (defined in corollary 42) and write in the same way its extension by zero to the brownian motion equivalent to δ and $\tilde{\delta}$. Moreover we checked in proposition 43 that $\tilde{\partial}^{(q)*} \tilde{\partial}^{(q)}$ is the number operator $\tilde{\Delta}$ having $\psi_{\underline{i}}$ as eigenvector of eigenvalue $|\underline{i}|$.

Lemma 52. *Assume $\rho(q, N) < 1$ and $|q|N < 1$.*

- (i) *If $\tilde{\eta}_\alpha = \frac{\alpha}{\alpha + \Delta}$ then $Rg(\tilde{\eta}_\alpha^2) \cap M_0 \subset D(\Delta)$.*
- (ii) *Non-commutative polynomials (in X_i 's) are a core for $\Delta|_{D(\Delta) \cap L^2(M_0)}$.*

Proof. (i) Recall $\Delta = \delta^* \delta$, δ extending $\partial^{(q)}$. Consider $\xi = \psi(x), x \in \mathcal{H}^{\otimes n}$, recall $\tilde{\partial}^{(q)}(\xi) = \partial(\xi) \# \Xi_q^{1/2}$ and $\partial^{(q)}(\xi) = \partial(\xi) \# \Xi_q$ and compute using lemma 31 (i) since ξ is a non-commutative polynomial and we get a bounded H (using lemmas 47, 49 and Theorem 50 to get assumption 2 (a),(b),(b'),(c) with $R = R_{\infty, \epsilon}$ any ϵ and note that for $\epsilon = \infty$ $x \in \mathcal{B} \subset D(\Delta^{1/2} \Delta_j)$ since $\tilde{\sigma} = 2$)

$$\begin{aligned}
\|\Delta \xi\|_2^2 &= \sum_i \langle \partial_i \Delta \xi, \partial_i^{(q)}(\xi) \# \Xi_q \rangle = \sum_i \langle \Delta \otimes 1 + 1 \otimes \Delta \partial_i \xi + H(\partial(\xi)), \partial_i^{(q)}(\xi) \# \Xi_q \rangle \\
&\leq \|H\| \|\tilde{\partial}^{(q)}(\xi)\|_2^2 \|\Xi_q^{3/2}\|_{M \otimes M^{op}} \|\Xi_q^{-1/2}\|_{M \otimes M^{op}} \\
&+ \sum_i \|\partial^{(q)} \otimes 1 \partial_i \xi\|_2 \|\partial^{(q)} \otimes 1 (\partial_i^{(q)}(\xi) \# \Xi_q)\|_2 + \sum_i \|1 \otimes \partial^{(q)} \partial_i \xi\|_2 \|1 \otimes \partial^{(q)}(\partial_i^{(q)}(\xi) \# \Xi_q)\|_2 \\
&=: (I) + (II) + (III) \\
(I) &\leq \|H\| n \|\xi\|_2^2 \|\Xi_q^{3/2}\|_{M \otimes M^{op}} \|\Xi_q^{-1/2}\|_{M \otimes M^{op}} \\
(II) &\leq \sum_i \|\Xi_q^{1/2}\|_{M \otimes M^{op}}^2 \|\Xi_q\|_{M \otimes M^{op}} \|\tilde{\partial}^{(q)} \otimes 1 \partial_i \xi\|_2^2 \\
&+ \|\Xi_q^{1/2}\|_{M \otimes M^{op}} \|\tilde{\partial}^{(q)} \otimes 1 \partial_i \xi\|_2 \|\partial_i(\xi) \# \partial^{(q)} \otimes 1 \Xi_q^2\|_2 \\
&\leq n \|\Xi_q^{1/2}\|_{M \otimes M^{op}}^2 \|\Xi_q^2\|_{M \otimes M^{op}} \|\partial \xi\|_2^2 + \|\Xi_q^{1/2}\|_{M \otimes M^{op}} \sqrt{n} \|\partial(\xi)\|_2^2 \left(\sum_j \|\partial_j^{(q)} \otimes 1 \Xi_q^2\|_{M \otimes M \otimes M^{op}}^2 \right)^{1/2} \\
\|\Delta \xi\|_2^2 &\leq \|H\| n \|\xi\|_2^2 \|\Xi_q^{3/2}\|_{M \otimes M^{op}} \|\Xi_q^{-1/2}\|_{M \otimes M^{op}} + 2n^2 \|\Xi_q^{1/2}\|_{M \otimes M^{op}}^2 \|\Xi_q^{-1/2}\|_{M \otimes M^{op}}^2 \|\Xi_q^2\|_{M \otimes M^{op}} \|\xi\|_2^2 \\
&+ \sqrt{n} \|\Xi_q^{-1/2}\|_{M \otimes M^{op}}^2 \|\Xi_q^{1/2}\|_{M \otimes M^{op}} n \|\xi\|_2^2 \left(\sum_j \|\partial_j^{(q)} \otimes 1 \Xi_q\|_{M \otimes M \otimes M^{op}}^2 \right)^{1/2} \leq C n^2 \|\xi\|_2^2
\end{aligned}$$

We mainly computed with derivation properties and equivalence of derivations. We also used the equality coming from $\tilde{\Delta}$ the number operator, using $\partial \xi$ is in the span of eigenvectors of number less than $n - 1$.

Thus consider $\tilde{\eta}_\alpha = \frac{\alpha}{\alpha + \Delta}$, take ξ a noncommutative polynomial in X_i 's so that so is $\tilde{\eta}_\alpha(x)$, let x_n its component on $\mathcal{H}^{\otimes n}$ so that we can bound and then extend by density to $x \in L^2$:

$$\begin{aligned}
\|\Delta \tilde{\eta}_\alpha^2(x)\|_2 &\leq \sum_{n>0} \left(\frac{\alpha}{\alpha+n}\right)^2 \|\Delta(x_n)\|_2 \leq \sum_{n>0} \left(\frac{\alpha}{\alpha+n}\right)^2 Cn \|x_n\|_2 \\
&\leq C \left(\sum_{n>0} \left(\frac{\alpha}{\alpha+n}\right)^4 n^2\right)^{1/2} \|x\|_2 \leq C\alpha^2 \left(\sum_{n>0} \frac{1}{n^2}\right)^{1/2} \|x\|_2 = C\alpha^2 \pi / \sqrt{6} \|x\|_2
\end{aligned}$$

- (ii) We want to prove that for $Z \in D(\Delta)$, $\Delta(\tilde{\eta}_\alpha^4(Z))$ converges weakly to $\Delta(Z)$. Since $\tilde{\eta}_\alpha^4(Z)$ converges to Z and Δ is closed, it suffices to bound $\|(\Delta(\tilde{\eta}_\alpha^4(Z)))\|_2$ uniformly in α . Then, the core property is obvious since $\tilde{\eta}_\alpha^4$ leave polynomials invariant and $\Delta \tilde{\eta}_\alpha^4$ bounded by (i), one first takes a convex combination such that $\|(\Delta(\sum \rho_n \tilde{\eta}_{\alpha_n}^4(Z) - Z))\|_2 \leq \epsilon/2$ $\|\sum \rho_n \tilde{\eta}_{\alpha_n}^4(Z) - Z\|_2 \leq \epsilon/2$ and then choose P polynomial such that $\|P - Z\|_2 \leq \epsilon/2 \|\Delta \circ \sum \rho_n \tilde{\eta}_{\alpha_n}^4\|$.

The main boundedness we thus need to prove will be based on the almost commutation of lemma 31 (v) applied under the assumptions stated in (iv). We apply it to δ given by $\partial \oplus \partial^{(q)}$ (of course ∂ is introduced here to check almost coassociativity more easily), $\delta_{(1)}$ given by $\partial^{(q)}$, $\delta_{(2)}$ given by ∂ , and $\delta_{(3)}$ by $\tilde{\partial}^{(q)}$. In that case, coassociativity assumptions are given by lemma 47. The other assumption in proposition 24 are given in lemma 49, the core property for $\tilde{\Delta}^{3/2}$ is obvious since non-commutative polynomials are a core for it (using the explicit non-commutative polynomial eigenvectors), the other assumptions comes from lemma 48.

As a consequence we have : $\tilde{\eta}_\alpha^\otimes \delta_i(Z) - \delta_i \tilde{\eta}_\alpha(Z) = \frac{1}{\alpha} \tilde{\eta}_\alpha^\otimes H_{i,\alpha}(Z)$ so that $\tilde{\eta}_\alpha^{\otimes 4} \delta_i(Z) - \delta_i \tilde{\eta}_\alpha^4(Z) = \sum_{j=0}^3 \frac{1}{\alpha} \tilde{\eta}_\alpha^{\otimes(1+j)} H_{i,\alpha} \tilde{\eta}_\alpha^{(3-j)}(Z)$.

We also have $H_{i,\alpha}$ bounded by $c_4^2 \alpha$ and $\|H_{i,\alpha}(x)\|_2 \leq c\sqrt{\alpha}(\|\delta(x)\|_2 + \|x\|_2)$

Proving first an adjoint commutation relation, one also deduces for any $Z \in D(\Delta)$: $\tilde{\eta}_\alpha^{\otimes 4} \Delta(Z) - \Delta \tilde{\eta}_\alpha^4(Z) = \frac{1}{\alpha} \sum_i \sum_{j=0}^3 (\tilde{\eta}_\alpha^\otimes \delta_i)^* \tilde{\eta}_\alpha^{\otimes(j)} H_{i,\alpha} \tilde{\eta}_\alpha^{(3-j)}(Z) - (\tilde{\eta}_\alpha^{\otimes(1+j)} H_{i,\alpha} \tilde{\eta}_\alpha^{(3-j)})^* (\delta(Z))$.

For our Z we are interested in, we thus deduce our concluding bound : $\|\Delta \tilde{\eta}_\alpha^4(Z)\|_2 \leq \|\Delta(Z)\|_2 + \frac{1}{\alpha} 4N \tilde{c}^2 \alpha (\|\delta(Z)\|_2 + \|Z\|_2)$.

□

Proposition 53. *For q such that $\rho(q, N) < 1$ (defined in corollary 42 for $|q|\sqrt{N} \leq 0.13$) and with moreover $|q|N < 1$, $\partial^{(q)}$ (giving δ), $\partial^{(q,Q)}$ (as giving $\delta_{(Q)}$) and the free difference quotient ∂ (giving $\tilde{\delta}$) satisfy assumption 2. The same is true of the derivation $\partial^{(q,U,\epsilon)} = \epsilon \partial^{(q)} \oplus \partial^{(q)} \# U$ (giving δ), $\partial^{(q,Q,U,\epsilon)} = \epsilon \partial^{(q,Q)} \oplus \partial^{(q,Q)} \# U$ (as giving $\delta_{(Q)}$) with a sum of two difference quotients $\partial \oplus \partial$ (giving $\tilde{\delta}$) for any $U \in (\mathcal{C}\langle X_1, \dots, X_N \rangle)^{\otimes_{alg} 2})^N$. As a consequence $\delta_0(X_1, \dots, X_N) = N$ under those conditions.*

Proof. Assumption 2(a),(b),(c) are checked in lemmas 47 and 49. (c') is checked in the previous lemma 52 (knowing lemma 49 to see non commutative polynomials are in \mathcal{B}). The analogue of lemma 52 in cases with U is similar and left to the reader.

Having checked assumption 2, we can now prove the consequence on microstate free entropy dimension. From corollary 38, applied to $\partial^{(q,U,\sqrt{t})}$ we get for any $1 \geq t \geq 0$, q -Gaussian variables $X_{1,t}, \dots, X_{n,t} \in \Gamma_q(\mathcal{H}) * L(F(\infty))$ and $S_1, \dots, S_N \in L(F(\infty))$ a free $(0, 1)$ -semicircular family, free from $\Gamma_q(\mathcal{H})$, with moreover, $X_{j,t,U} \in W^*(X_1, \dots, X_N, S_1, \dots, S_N, \{S'_j\}_{j=0}^\infty)$ where $\{S'_j\}_{j=0}^\infty$ is a free semicircular family free with $\{X_1, \dots, X_n, S_1, \dots, S_N\}$.

and such that

$$\|X_{j,t,U} - X_j - \sqrt{t}\overline{\partial_j^{(q)}}(X_j) \# U_j \# S_j\|_2 \leq (c + d) t,$$

where $c^2 = \frac{1}{4}(\|\partial^{(q,U,\sqrt{t})} \partial^{(q,U,\sqrt{t})}(X_j)\|_2 + \sqrt{2}\|(\partial^{(q,U,\sqrt{t})} \otimes 1 \oplus 1 \otimes \partial^{(q,U,\sqrt{t})})\partial^{(q,U,\sqrt{t})}(X_j)\|_2)^2$ and $d = \|\Xi_q\|_2$. Since $\partial^{(q)*}$ extends to a bounded map on $M \hat{\otimes} M$ (say with bound C), $c \leq C\|\Xi_q\|_{M \hat{\otimes} M}(1 + \|U_j^* U_j\|_{M \hat{\otimes} M}) + 2(\sup_i \|U_i\|_{M \overline{\otimes} M} + 1)(2\|(\partial^{(q)} \otimes 1 \oplus 1 \otimes \partial^{(q)})\Xi_q\|_2 + \|(\partial^{(q)} \otimes 1 \oplus 1 \otimes \partial^{(q)})U_j\|_2)$.

Now let us take $U_j = \Xi_{p,n} := \sum_{k=0}^n (1 \otimes 1 - \Xi_q^p)^k$ powers being taken in $M \otimes M^{op}$, with $\Xi_q^p = \sum_{k=0}^p q^k \sum_{\xi \text{ b. o. n.}} \mathcal{H}^{\otimes k} \xi \otimes \xi^*$ the truncated (polynomial) variant of Ξ_q , so that $\|U_j\|_{M \hat{\otimes} M} \leq \frac{\nu(q,N)^{n+1}-1}{\nu(q,N)-1}$, using also the bound above for $\|\partial_k \otimes 1 \Xi_q\|_2$, one easily gets in the range of q we consider constants A, B, R such that $c + d$ above is below $A + B R^{n+1}$ uniformly in p, n . Moreover $\Xi_q \# \Xi_{p,n} - 1 \otimes 1 = (\Xi_q - \Xi_q^p) \# \Xi_{p,n} - (1 \otimes 1 - \Xi_q^p)^{n+1}$ so that $\|\Xi_q \# \Xi_{p,n} - 1 \otimes 1\|_{L^2(M \overline{\otimes} M^{op})} \leq \|(\Xi_q - \Xi_q^p)\|_{L^2(M \overline{\otimes} M^{op})} \|\Xi_{p,n}\|_{M \overline{\otimes} M^{op}} + \|(1 \otimes 1 - \Xi_q^p)\|_{M \overline{\otimes} M^{op}}^{n+1} \leq \frac{(q\sqrt{N})^{p+1}}{(1-(q\sqrt{N}))(1-\rho)} + \rho^{n+1}$, using the estimate and notation of Corollary 42. Taking $p = n$ and since $\rho \geq |q|\sqrt{N}$ we got another constant D with $(X_{j,t,n} = X_{j,t,\Xi_{n,n}})$:

$$\|X_{j,t,n} - X_j + \sqrt{t}S_j\|_2 \leq (A + BR^{n+1}) t + \sqrt{t}D\rho^{n+1},$$

Since $\rho < 1, R \geq 1$, choosing l large enough integer to get $1/l < \log(1/\rho)/\log(R)$ so that if $t^{1/2(l+1)} = \rho^{n+1}$ then $R^{n+1} = (\rho R^{1/l})^{(n+1)l} t^{-l/2(l+1)} \leq t^{-l/2(l+1)}$ so that choosing $t \rightarrow 0$ as above when $n \rightarrow \infty$ we have a sequence satisfying $\|X_{j,t,n} - X_j - \sqrt{t}S_j\|_2 \leq At + (B + D)t^{1/2+1/2(l+1)}$. From the proof of [46, Theorem 4] (in which we see the upper bound in At is not necessary, having only a $At^\alpha, \alpha > 1/2$, as we have here, is enough), knowing R^ω -embeddability from [47], one deduces the result on microstate free entropy dimension. \square

4.5. Group Cocycles. Since assumption 1 is hard to verify in practice, it is interesting to work only under assumption 0, and prove directly that the ultramild solution of theorem 10 (i) satisfy $\|X_t\|_2 = \|X_0\|_2$ a.e. to get a stationary solution. In this part, we find a necessary and sufficient condition for derivations coming from group cocycles to get results in the spirit of Corollary 3 in [46].

Let Γ be a discrete group. To a (n additive left) cocycle c with value in the regular representation $c \in C^1(\Gamma, \ell^2(\Gamma))$ we associate a derivation $\delta_c : \mathbb{C}\Gamma \rightarrow \ell^2(\Gamma) \otimes \ell^2(\Gamma) = L^2(M_0 \otimes M_0)$ (M_0 the group von Neumann algebra of Γ) given by $\delta_c(\gamma) = B(c(\gamma))\gamma$ where $B : \ell^2(\Gamma) \rightarrow \ell^2(\Gamma) \otimes \ell^2(\Gamma)$ the isometry given by $B(\gamma) = \gamma \otimes \gamma^{-1}$. Indeed, $\delta(\gamma_1 \gamma_2) = B(\gamma_1 c(\gamma_2) + c(\gamma_1))\gamma_1 \gamma_2 = \gamma_1 B(c(\gamma_2))\gamma_1^{-1} \gamma_1 \gamma_2 + \delta_c(\gamma_1)\gamma_2 = \gamma_1 \delta_c(\gamma_2) + \delta_c(\gamma_1)\gamma_2$ so that δ_c is a derivation with the same bimodule structure used earlier on $L^2(M_0 \otimes M_0)$. Moreover δ_c is easily seen to be a real derivation if c takes values in $i\mathbb{R}$ (we will consider only such cocycles). Let us note that $\langle \delta_c(\gamma), 1 \otimes 1 \rangle = 0$ for any gamma so that we easily deduce that $\delta_c^*(1 \otimes 1) = 0$ so that δ_c is always closable. Any $\delta_{c_1}, \dots, \delta_{c_n}$ therefore satisfy assumption 0. Moreover, as noted e.g. in the proof of Corollary 3 in [46], $\langle \delta_c \gamma, \delta_c \gamma' \rangle = \delta_\gamma' \|c(\gamma)\|_2^2$ so that $\delta_c^* \delta_c(\gamma) = \|c(\gamma)\|_2^2 \gamma$. We now fix c_1, \dots, c_n such cocycles and note δ_i the extension to M of δ_{c_i} described at the beginning of section 2. We write X_t, X_t^ϵ the ultramild (resp mild) solution given by theorem 10 when the initial condition is X_0 .

We now want to describe a first equivalent formulation of the isometry $\|X_t\|_2 = \|X_0\|_2$. To this end, we want to give an equation on certain component of the free product $L^2(M)$. Note

N the von Neumann algebra generated by free Brownian motions, it is well known that M is the orthogonal direct sum of $L^2(N)$ and $L^2(N)\gamma_1(L^2(N)\ominus\mathbb{C})\gamma_2\cdots(L^2(N)\ominus\mathbb{C})\gamma_n L^2(N)$ where γ_i 's run over $\Gamma - \{1\}$. Since X_t and X_t^ϵ are orthogonal to $L^2(N)$ (since $\delta_i^* 1 \otimes 1 = 0$) we may consider only $X_{t;\gamma_1,\dots,\gamma_n}^\epsilon \in L^2(N) \otimes (L^2(N) \ominus \mathbb{C})^{n-1} \otimes L^2(N)$ such that $X_{t;\gamma_1,\dots,\gamma_n}^\epsilon \# (\gamma_1 \otimes \dots \otimes \gamma_n)$ are the orthogonal projections on those spaces. We wrote here $U \# (\gamma_1 \otimes \dots \otimes \gamma_n)$ the extension given by freeness of $(a_1 \otimes \dots \otimes a_{n+1}) \# (\gamma_1 \otimes \dots \otimes \gamma_n) = a_1 \gamma_1 a_2 \dots a_n \gamma_n a_{n+1}$. We now have the following :

Proposition 54. *Assume $X_0 = \gamma$, then :*

$$\begin{aligned} X_{t;\gamma_1,\dots,\gamma_n}^\epsilon &= \delta_{n=1} \delta_{\gamma_1=\gamma} e^{-\frac{t}{2}(\sum_{j=1}^N \|c_j(\gamma)\|_2^2)} 1 \otimes 1 \\ &+ (1 - \epsilon) \sum_{i=1}^n \sum_{j=1}^N \\ &\int_0^t e^{\frac{s-t}{2}(\sum_{i=1}^n \sum_{j=1}^N \|c_j(\gamma_i)\|_2^2)} X_{s;\gamma_1,\dots,\gamma_n}^\epsilon \#_i (\langle \gamma_i, c_j(\gamma_i) \rangle 1 \otimes dS_s^{(j)} + \langle 1, c_j(\gamma_i) \rangle dS_s^{(j)} \otimes 1) \\ &+ (1 - \epsilon) \delta_{n \neq 1} \sum_{i=1}^{n-1} \sum_{j=1}^N \langle \gamma_i, c_j(\gamma_i \gamma_{i+1}) \rangle \times \\ &\times \int_0^t e^{\frac{s-t}{2}(\sum_{i=1}^n \sum_{j=1}^N \|c_j(\gamma_i)\|_2^2)} X_{s;\gamma_1,\dots,\gamma_i \gamma_{i+1},\dots,\gamma_n}^\epsilon \#_i 1 \otimes dS_s^{(j)} \otimes 1, \end{aligned}$$

which is non zero only if $\gamma_1 \dots \gamma_n = \gamma$. (We have noted $a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \dots \otimes a_n \#_i 1 \otimes (S_t - S_s) \otimes 1 = a_1 \otimes \dots \otimes a_i \otimes (S_t - S_s) \otimes a_{i+1} \dots \otimes a_n$, $a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \dots \otimes a_n \#_i (S_t - S_s) \otimes 1 = a_1 \otimes \dots \otimes a_i (S_t - S_s) \otimes a_{i+1} \dots \otimes a_n$, $a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \dots \otimes a_n \#_i 1 \otimes (S_t - S_s) = a_1 \otimes \dots \otimes a_i \otimes (S_t - S_s) a_{i+1} \dots \otimes a_n$ and the evident corresponding adapted stochastic integrals.) Moreover this relation with $\epsilon = 0$ is thus also valid for X_t (by the weak convergence defining it). ■

As a consequence, using freeness and the definition of the space where $X_{t;\gamma_1,\dots,\gamma_n}^\epsilon$ lives (especially the orthogonal complements to \mathbb{C}) we get :

$$\begin{aligned} \|X_{t;\gamma_1,\dots,\gamma_n}\|_2^2 &= \delta_{n=1} \delta_{\gamma_1=\gamma} e^{-t(\sum_{j=1}^N \|c_j(\gamma)\|_2^2)} \\ &+ \sum_{i=1}^n \sum_{j=1}^N (|\langle \gamma_i, c_j(\gamma_i) \rangle|^2 + |\langle 1, c_j(\gamma_i) \rangle|^2) \int_0^t ds e^{(s-t)(\sum_{i=1}^n \sum_{j=1}^N \|c_j(\gamma_i)\|_2^2)} \|X_{s;\gamma_1,\dots,\gamma_n}\|_2^2 \\ &+ \delta_{n \neq 1} \sum_{i=1}^{n-1} \sum_{j=1}^N |\langle \gamma_i, c_j(\gamma_i \gamma_{i+1}) \rangle|^2 \int_0^t ds e^{(s-t)(\sum_{i=1}^n \sum_{j=1}^N \|c_j(\gamma_i)\|_2^2)} \|X_{s;\gamma_1,\dots,\gamma_i \gamma_{i+1},\dots,\gamma_n}\|_2^2. \end{aligned}$$

As a consequence, solving the equation by variation of constants, and using the following convenient notation $\|\hat{c}_j(\gamma)\|_2^2 = \|c_j(\gamma)\|_2^2 - (|\langle \gamma_i, c_j(\gamma_i) \rangle|^2 + |\langle 1, c_j(\gamma_i) \rangle|^2)$, we obtain the following :

Proposition 55. *Assume $X_0 = \gamma$, then*

$$\begin{aligned} \|X_{t;\gamma_1,\dots,\gamma_n}\|_2^2 &= \delta_{n=1} \delta_{\gamma_1=\gamma} e^{-t(\sum_{j=1}^N \|\hat{c}_j(\gamma)\|_2^2)} \\ &+ \delta_{n \neq 1} \sum_{i=1}^{n-1} \sum_{j=1}^N |\langle \gamma_i, c_j(\gamma_i \gamma_{i+1}) \rangle|^2 \int_0^t ds e^{(s-t)(\sum_{i=1}^n \sum_{j=1}^N \|\hat{c}_j(\gamma_i)\|_2^2)} \|X_{s;\gamma_1,\dots,\gamma_i \gamma_{i+1},\dots,\gamma_n}^\epsilon\|_2^2. \end{aligned}$$

■

This equation is nothing but a forward Kolmogorov equation, and the question we ask is whether $1 = \|\gamma\|_2^2 = \|X_0\|_2^2 = \sum_n \sum_{\gamma_1,\dots,\gamma_n} \|X_{t;\gamma_1,\dots,\gamma_n}\|_2^2$, i.e. nothing but if the solution of the Kolmogorov equation is conservative. In order to state a result, let us define a corresponding continuous time Markov chain to give a probabilistic counterpart to the stationarity of X_t , using usual results on Kolmogorov equations (cf. e.g. [25]).

Notation 56. *Given a countable group Γ and additive left cocycles with value in the left regular representation c_1, \dots, c_N as above. We write $M(\Gamma; c_1, \dots, c_N)$ the continuous time Markov process defined on the countable state space $F(\Gamma) = (\Gamma - \{1\})^{(<\omega)}$ defined by the following rates $R((\gamma_1, \dots, \gamma_n)) = \sum_{i=1}^n \sum_{j=1}^N \|\hat{c}_j(\gamma_i)\|_2^2$, and with transition probabilities non zero only from $(\gamma_1, \dots, \gamma_n)$ to $(\gamma_1, \dots, \delta_i, \delta'_i, \dots, \gamma_n)$ with $\delta_i \delta'_i = \gamma_i$ (of course $\delta_i, \delta'_i \neq 1$), given by*

$$P((\gamma_1, \dots, \gamma_n), (\gamma_1, \dots, \delta_i, \delta'_i, \dots, \gamma_n)) = \frac{\sum_{j=1}^N |\langle \delta_i, c_j(\gamma_i) \rangle|^2}{R((\gamma_1, \dots, \gamma_n))}.$$

We can now state the following trivial :

Corollary 57. *Let X_t be the ultramild solution given by theorem 10 with $\delta = (\delta_1, \dots, \delta_N)$ associated as above to cocycles (c_1, \dots, c_N) . Then $\|X_t\|_2 = \|X_0\|_2$ (for any $X_0 \in \ell^2(\Gamma)$) for all $t \in [0, T)$ (and as a consequence is stationary in $[0, T)$ on $M_0 = L(\Gamma)$) if and only if $M(\Gamma; c_1, \dots, c_N)$ has almost surely no explosion before T .* ■

REFERENCES

- [1] P. BIANE, M. CAPITAINE and A. GUIONNET. Large deviation bounds for matrix brownian motion. *Invent. Math.*, 152:433–459, 2003.
- [2] P. BIANE and R. SPEICHER. Stochastic calculus with respect to free Brownian Motion. *Probability Theory and related Fields*, 112, no.3:373–409, 1998.
- [3] P. BIANE and R. SPEICHER. Free diffusions, free entropy and free Fisher information. *Ann. Inst. H. Poincaré, Probabilités et Statistiques, PR*, 37:581–606, 2001.
- [4] P. BIANE and D. VOICULESCU. A Free Probability Analogue of the Wasserstein Metric on the Trace-State Space. *Geom. Func. Anal.*, 11:1125–1138, 2001.
- [5] M. BOŽEJKO, Ultracontractivity and strong Sobolev inequality for q -Ornstein-Uhlenbeck semigroup ($-1 < q < 1$), *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 2:203–220, 1998.
- [6] M. BOŽEJKO, B. KUMMERER, R. SPEICHER, q -Gaussian processes : Non-commutative and classical aspects, *Commun. Math. Phys.* 185:129–154, 1997.
- [7] M. BOŽEJKO and R. SPEICHER, An example of a generalized Brownian motion, *Commun. Math. Phys.* 137:519 – 531, 1991.
- [8] A.M. CHEBORATEV and F. FAGNOLA. Sufficient Conditions for Conservativity of Minimal Quantum Dynamical Semigroups. *Journal of Functional Analysis*, 153(2):382–404, 1998.
- [9] F. CIPRIANI et J.-L. SAUVAGEOT. Derivations as square roots of Dirichlet forms. *Journal of Functional Analysis*, 201:78–120, 2003.

- [10] A. CONNES and D. SHLYAKHTENKO. L^2 -homology for von Neumann algebras. *J. Reine Angew. Math.*, 586:125–168, 2005.
- [11] Y. DABROWSKI. A note about proving non- Γ under a finite non-microstates free Fisher information Assumption. *Journal of Functional Analysis*, 258:3662–3674, 2008.
- [12] Y. DABROWSKI. A non-commutative Path Space approach to stationary free Stochastic Differential Equations. *arxiv*, OA:1006.4351, preprint 2010.
- [13] G. DA PRATO and J. ZABCZYK. *Stochastic equations in Infinite Dimensions*. Cambridge University Press, 1992.
- [14] E.B. DAVIS and J.M. LINDSAY. Non-commutative symmetric Markov semigroups. *Math. Zeitschrift*, 210:379–411, 1992.
- [15] C. DONATI-MARTIN. Stochastic integration with respect to q Brownian motion. *Probab. Theory Relat. Fields*, 125:77–95, 2003.
- [16] K. DYKEMA and A. NICA, On the Fock representation of the q -commutation relations, *J. Reine Angew. Math.* 440:201–212, 1993.
- [17] F. FAGNOLA. H-P Quantum Stochastic Differential Equations. In *NON-COMMUTATIVITY, INFINITE-DIMENSIONALITY, AND PROBABILITY AT THE CROSSROADS The Proceedings of the RIMS Workshop on Infinite Dimensional Analysis and Quantum Probability*, pages 51–96, November 2001.
- [18] F. FAGNOLA and S. J. WILLS. Mild Solutions of Quantum Stochastic Differential Equations. *Electronic Communications in Probability*, 5:158–171, 2000.
- [19] L. GE. Applications of free entropy to finite von Neumann algebras II.. *Annals of Mathematics*, 147:143–157, 1998.
- [20] A. GUIONNET and D. SHLYAKHTENKO. Free Diffusions and Matrix Models with Strictly Convex Interaction. *Geom. Func. Anal.*, 18:1875–1916, 2007.
- [21] T. KATO. *Perturbation Theory for Linear Operators*. Springer second edition, 1980.
- [22] T. KOTELNEZ. *Stochastic ordinary and stochastic partial differential equations: transition from microscopic to macroscopic equations*. Springer 2008.
- [23] N.V. KRYLOV. An analytic approach to SPDEs. In R.A. Carmona B.L. Rozovskii (Ed.) *Stochastic Partial differential equations: Six perspectives*, AMS Mathematical Surveys and Monographs Vol 64, 1999.
- [24] N.V. KRYLOV and B.L. ROZOVSKII. Stochastic evolution equations. *Current problems in mathematics*, Vol. 14 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1979, pp. 71–147, 256.
- [25] T. M. LIGGETT. *Continuous Time Markov Processes: An Introduction*. AMS, 2010.
- [26] W. LÜCK. Dimension theory of arbitrary modules over finite von neumann algebras and L^2 -beti numbers I: foundations. *J. Reine Angew. Math*, 495:135–162, 1998.
- [27] W. LÜCK. *L^2 -Invariants : Theory and Applications to Geometry and K-Theory*. Springer, 2002.
- [28] Z.M. MA and M RÖCKNER. *Introduction to the theory of (non-symmetric) Dirichlet forms*. Universitext. Springer, Berlin, 1992.
- [29] I. MINEYEV and D. SHLYAKHTENKO. Non-microstate Free Entropy Dimension for Groups. *Geom. Func. Anal.*, 15:476–490, 2005.
- [30] A. NOU. Non injectivity of the q -deformed von Neumann algebra. *Math. Ann.* 330(1):17–38, 2004.
- [31] F. OTTO and C. VILLANI. Generalization of an Inequality by Talagrand and Links with the Logarithmic Sobolev Inequality. *Journal of Functional Analysis*, 173:361–400, 2000. Solid von Neumann algebras. *Acta Math.* 192 (2004), no. 1, 111–117.
- [32] N. OZAWA. Note on closable derivations on a finite von Neumann Algebra. (unpublished), cf. <http://sitemason.vanderbilt.edu/files/iOHBuw/ozawa.pdf>.
- [33] E. PARDOUX. Sur des équations aux dérivées partielles stochastiques monotones. *C. R. Acad. Sci. Paris Sér. A-B* 275:A101–A103, 1972.
- [34] E. PARDOUX. Équations aux dérivées partielles stochastiques de type monotone, *Séminaire sur les Équations aux Dérivées Partielles (1974–1975)*, III Exp. No. 2, Collège de France, Paris, 1975, p. 10.

- [35] J. PETERSON. A 1-cohomology characterisation of Property (T) in von Neumann algebras. *Pacific Journal of Mathematics*, 243:181–199, 2009.
- [36] J. PETERSON. L^2 -rigidity in von Neumann algebras. *Inven. Math.* 175:417–433, 2009.
- [37] C. PRÉVÔT and M. RÖCKNER. *A Concise Course on Stochastic Partial Differential Equations*. Springer, Lecture Notes In Mathematics 1905, 2007.
- [38] B.L. ROZOVSKII. *Stochastic Evolution Systems*. Kluwer Academic, Dordrecht/Norwell, 1990.
- [39] S. SAKAI. *C^* -algebras and W^* -algebras*. Springer, 1971.
- [40] J.L. SAUVAGEOT. Markov quantum semigroups admit covariant Markov C^* -dilations. *Comm. Math. Phys.*, 106, 1986.
- [41] J.-L. SAUVAGEOT. Tangent bimodules and locality for dissipative operators on C^* -algebras. *Quantum Probability and applications*, IV:322–338, 1989. Lecture notes in math 1396.
- [42] J.-L. SAUVAGEOT. Quantum Dirichlet forms, Differential Calculus and Semigroups. *Quantum Probability and Applications V, Lecture Notes in Mathematics*, 1442:334–346, 1990.
- [43] J.-L. SAUVAGEOT. Strong Feller semigroups on C^* -algebras. *Journal of Operator Theory* 42:83–102, 1999.
- [44] D. SHLYAKHTENKO. Some estimates for non-microstate free entropy dimension with applications to q -semicircular families. *International Mathematics Research Notices*, 51:2757–2772, 2004.
- [45] D. SHLYAKHTENKO. Remarks on free entropy dimension. In *Operator Algebras Abel Symposia, Volume 1*. Springer, 249–257, 2006. .
- [46] D. SHLYAKHTENKO. Lower estimates on microstate free entropy dimension. *Analysis and PDE*, 2:119–146, 2009.
- [47] P. ŚNIADY, *Gaussian random matrix models for q -deformed Gaussian variables*, *Comm. Math. Phys.*, 216(3):515–537, 2001.
- [48] P. ŚNIADY, *Factoriality of Bożejko-Speicher von Neumann algebras*, *Comm. Math. Phys.*, 246(3):561–567, 2004.
- [49] M. TAKESAKI. *Theory of Operator Algebras*. Encyclopaedia of Mathematical Sciences Vol 124, 126, 127, Springer 2003.
- [50] A.S. USTUNEL. On the regularity of the solutions of stochastic partial differential equations. *Stochastic Differential Systems Filtering and Control* Lecture Notes in Control and Information Sciences, 69:71–75 1985.
- [51] D. VOICULESCU. The analogs of entropy and of Fisher’s information measure in free probability theory, II. *Invent. Math.* 118(3):411–440, 1994.
- [52] D. VOICULESCU. The analogs of entropy and of Fisher’s information measure in free probability theory, III : The absence of Cartan subalgebras. *Geometric and Functional Analysis* 6(1):172–199, 1986.
- [53] D. VOICULESCU. The analogs of entropy and of Fisher’s information measure in free probability theory, V : Non commutative Hilbert Transforms. *Inventiones mathematicae*, 132:189–227, 1998.
- [54] D. VOICULESCU. The analogs of entropy and of Fisher’s information measure in free probability theory, VI : Liberation and Mutual free Information. *Advances in Mathematics*, 146:101–166, 1999.
- [55] D. VOICULESCU. Free Entropy. *Bulletin of the London Mathematical Society*, 34(3):257–278, 2002.
- [56] J.B. WALSH. An Introduction to Stochastic Partial Differential Equations, In *École d’été de probabilités de Saint-Flour, XIV–1984*, Lecture Notes in Math., vol. 1180, Springer, Berlin, pp. 265–439.
- [57] N. WEAVER. Lipschitz algebras and derivations of von Neumann algebras. *Journal of Functional Analysis*, 139:261–300, 1996.
- [58] D. WILLET. A Linear generalization of Gronwall’s inequality. *Proceedings of the American Mathematical Society*, 16, No. 4:774–778, 1975.
- [59] D. ZAGIER. Realizability of a model in infinite statistics,. *Commun. Math. Phys.*, 147:199–210, 1992.

UNIVERSITÉ DE LYON, UNIVERSITÉ LYON 1, INSTITUT CAMILLE JORDAN, 43 BLVD. DU 11 NOVEMBRE 1918, F-69622 VILLEURBANNE CEDEX, FRANCE

E-mail address: dabrowski@math.univ-lyon1.fr